ON THE PROJECTIVE AND CONFORMAL TRANSFORMATIONS IN THE METRIC MANIFOLD WITH TORSION

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§ 1. Introduction

We consider an n-dimensional manifold on which there is given a positive metric

$$ds^2 = g_{jk} dx^j dx^k,$$

and take the quantities E^i_{jk}, so that

$$\frac{\partial g_{jk}}{\partial x^l} - g_{sk} E^s_{jl} - g_{js} E^s_{kl} = 0. \tag{1.1}$$

Then for the coordinate transformation

$$x'^i = x'^i(x^1, \cdots\cdots, x^n)$$

of which the Jacobian is different from zero, we can calculate

$$\frac{\partial x^a}{\partial x'^i} E'^i_{jk} = \frac{\partial^2 x^a}{\partial x'^j \partial x'^k} + \frac{\partial x^b}{\partial x'^j} \frac{\partial x^c}{\partial x'^k} E^a_{bc}, \tag{1.2}$$

and consequently, E^i_{jk} are the components of metric connection, where we assume that E^i_{jk} needs not to be symmetric, i. e., $E^i_{jk} \neq E^i_{kj}$.

If we put

$$S_{jk}{}^i = \frac{1}{2}(E^i_{jk} - E^i_{kj}), \tag{1.3}$$

then we know evidently that $S_{jk}{}^i$ are the components of tensor being anti-symmetric in j and k, and we call it a torsion tensor and the metric manifold with such a connection E^i_{jk} the metric manifold with torsion.

From (1.1), we can calculate

$$E^i_{jk} = \left\{ {i \atop jk} \right\} + S_{jk}{}^i - S^i{}_{jk} - S^i{}_{kj}, \tag{1.4}$$

where

(1. 5) $$S^{i}{}_{jk}=g^{is}g_{kt}S_{sj}{}^{t},$$

and from (1.4), we have

(1. 6) $$\Gamma^{i}_{jk}=\frac{1}{2}(E^{i}_{jk}+E^{i}_{kj})=\left\{{i \atop jk}\right\}-S^{i}{}_{jk}-S^{i}{}_{kj},$$

since $S_{jk}{}^{i}$ is anti-symmetric in j and k. From (1.4), putting

(1. 7) $$\begin{cases} E^{i}_{jk}=\left\{{i \atop jk}\right\}+L_{jk}{}^{i}, \\ L_{jk}{}^{i}=S_{jk}{}^{i}-T^{i}{}_{jk}, \quad T^{i}{}_{jk}=S^{i}{}_{jk}+S^{i}{}_{kj}, \end{cases}$$

then (1.6) is reduced to

(1. 8) $$\Gamma^{i}_{jk}=\left\{{i \atop jk}\right\}-T^{i}{}_{jk},$$

and we find

(1. 9) $$E^{i}_{jkl}=R^{i}_{jkl}+L_{jk}{}^{i}{}_{,l}-L_{jl}{}^{i}{}_{,k}+L_{jk}{}^{s}L_{sl}{}^{i}-L_{jl}{}^{s}L_{sk}{}^{i},$$

where the comma denotes covariant differentiation with respect to $\left\{{i \atop jk}\right\}$,

(1.10) $$E^{i}_{jkl}=\frac{\partial}{\partial x^{l}}E^{i}_{jk}-\frac{\partial}{\partial x^{k}}E^{i}_{jl}+E^{s}_{jk}E^{i}_{sl}-E^{s}_{jl}E^{i}_{sk},$$

and R^{i}_{jkl} is the Riemannian curvature tensor. (*)

§ 2. The coordinate transformations

From (1.4), contracting for i and j, we have

$$E^{a}_{ak}=\left\{{a \atop ak}\right\}+S_{ak}{}^{a}-S^{a}{}_{ka},$$

since $S^{a}{}_{ak}=0$ from (1.5). Because of which

$$\left\{{a \atop ak}\right\}=\sum_{a=1}^{n}\frac{\partial}{\partial x^{k}}\log\sqrt{g_{aa}}=\frac{\partial}{\partial x^{k}}\sum_{a=1}^{n}\log\sqrt{g_{aa}},$$

then we obtain

$$E^{a}_{ak}=\frac{\partial}{\partial x^{k}}\sum_{a=1}^{n}\log\sqrt{g_{aa}}+S_{ak}{}^{a}-S^{a}{}_{ka},$$

(*) Here, we have refered from (1) for the preparation.

and consequently, E^a_{ak} are the components of the covariant vector. On the other hand, from (1.2), it can be reduced to

$$E'^i_{jk}=\frac{\partial x'^i}{\partial x^a}\Big(\frac{\partial^2 x^a}{\partial x'^j\partial x'^k}+\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}E^a_{bc}\Big),$$

and contracting for i and j, we obtain

$$E'^m_{mk}=\frac{\partial}{\partial x'^k}\log\Big|\frac{\partial x'}{\partial x}\Big|+\frac{\partial x^c}{\partial x'^k}E^a_{ac},$$

and finally, since E^a_{ac} are the components of a vector, we have

$$\frac{\partial}{\partial x'^k}\log\Big|\frac{\partial x'}{\partial x}\Big|=0,$$

and it implies the Jacobian be constant. Hence, we have the following theorem:

THEOREM *In the metric manifold with torsion, the Jacobian is constant for the general coordinate transformation.*

§3. The projective transformations

In this section, we investigate the relations between the metric manifold with torsion and the Riemannian manifold with the same metric for the projective transformations.

In the metric manifold with torsion, we consider the paths defined dy

$$\frac{d^2x^i}{ds^2}+E^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{3.1}$$

Interchanging j and k in (3.1), summing them and dividing it by 2, we obtain

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{3.2}$$

and conversely. In fact, from this, i.e.,

$$2\frac{d^2x^i}{ds^2}+(E^i_{jk}+E^i_{kj})\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

we have (3.1). Hence we can aware that the paths (3.1) are equivalent to (3.2), moreover Γ^i_{jk} is also a connection, and thus we can take (3.2) as the equation of paths instead of (3.1).

Now, we have known already that the necessary and sufficient condition that, on the metric manifolds V^n and $\bar{V}^n$ with torsion of which their metric tensors are g_{jk} and $\bar{g}_{jk}$ respectively, they are projectively transformable one another is that (3.3)

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\phi_k+\delta^i_k\phi_j.$$

For the projective transformation the Weyl tensor $W(\Gamma)^i_{jkl}$ is invariant [2], where

$$(3.4)\qquad W(\Gamma)^i_{jkl}=\Gamma^i_{jkl}+\frac{2}{n+1}\delta^i_j(\Gamma_{kl}-\Gamma_{lk})+\frac{1}{n^2-1}\,[\delta^i_k(n\Gamma_{jl}+\Gamma_{lj})-\delta^i_l(n\Gamma_{jk}+\Gamma_{kj})],$$

$$(3.5)\qquad \Gamma^i_{jkl}=\frac{\partial}{\partial x^l}\Gamma^i_{jk}-\frac{\partial}{\partial x^k}\Gamma^i_{jl}+\Gamma^s_{jk}\Gamma^i_{sl}-\Gamma^s_{jl}\Gamma^j_{sk}$$

$$(3.6)\qquad \Gamma_{jk}=\Gamma^i_{jki}.$$

Substituting (1.6) into (3.5), and putting

$$(3.7)\qquad T^i{}_{jk}=S^i{}_{jk}+S^i{}_{kj},$$

then we can have

$$(3.8)\qquad \Gamma^i_{jkl}=R^i_{jkl}-T^i_{jkl},$$

where

$$(3.9)\qquad T^i_{jkl}=T^i{}_{jk,l}-T^i{}_{jl,k}-T^s{}_{jk}T^i{}_{sl}+T^s{}_{jl}T^i{}_{sk}.$$

Contracting for i and l, we obtain $\Gamma_{jk}=R_{jk}-T_{jk}$, where $T_{jk}=T^i{}_{jki}$. Thus, since the Weyl tensor $W(\Gamma)^i_{jkl}$ is invariant for the projective transformations, it is reducible to

$$(3.10)\qquad W(\Gamma)^i_{jkl}=W^i_{jkl}-W(T)^i_{jkl},$$

where $W^i{}_{jkl}$ is the Weyl tensor on the Riemannian manifold, i.e.,

$$(3.11)\qquad W^i_{jkl}=R^i_{jkl}+\frac{1}{n-1}(\delta^i_kR_{jl}-\delta^i_lR_{jk}),$$

since R_{jk} is symmetric [3], and

$$W(T)^i_{jkl}=T^i_{jkl}+\frac{1}{n+1}\delta^i_j(T_{kl}-T_{lk})+\frac{1}{n^2-1}[\delta^i_k(nT_{jl}+T_{lj}) -\delta^i_l(nT_{jk}+T_{kj})]. \tag{3.12}$$

As the result of (3.10), when the Riemannian manifolds with the same metric as on the metric manifolds are projectively transformable, $W(T)^i_{jkl}$ is invariant, provided that the metric manifolds are also projectively transformable.

Now, analogically on the Riemannian manifold [3], it can be safely said the metric manifold with torsion be projectively flat, when $W(\Gamma)^i_{jkl}=0$, and hence we have the following theorem:

THEOREM *The necessary and sufficient condition that the metric manifold with torsion be projectively flat is that*

$$W^i_{jkl}=W(T)^i_{jkl}.$$

In the Riemannian manifold, the equation of paths, i.e., the geodesics are expressed by

$$\frac{d^2x^i}{ds^2}+\left\{{i \atop jk}\right\}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

and in the metric manifold with torsion, it is (3.2). The equation of the projective transformation on the Riemann manifold is represented by

$$\overline{\left\{{i \atop jk}\right\}}=\left\{{i \atop jk}\right\}+\delta^i_j\phi_k+\delta^i_k\phi_j.$$

From (1.6), this equation is reduced to

$$\bar{\Gamma}^i_{jk}+\bar{S}^i{}_{jk}+\bar{S}^i{}_{kj}=\Gamma^i_{jk}+S^i{}_{jk}+S^i{}_{kj}+\delta^i_j\phi_k+\delta^i_k\phi_j.$$

Contracting for i and j, making use of $S^a{}_{ak}=0$ and adjusting for $\phi,_k$ and substituting it into above form, then we obtain the form

$$\bar{\Gamma}^i_{jk}+\bar{S}^i{}_{jk}+\bar{S}^i{}_{kj}-\frac{1}{n+1}(\delta^i_j\bar{\Gamma}^a_{ak}+\delta^i_k\bar{\Gamma}^a_{aj})-\frac{1}{n+1}(\delta^i_j\bar{S}^a{}_{ka}+\delta^i_k\bar{S}^a{}_{ja}) \tag{3.13}$$
$$=\Gamma^i_{jk}+S^i{}_{jk}+S^i{}_{kj}-\frac{1}{n+1}(\delta^i_j\Gamma^a_{ak}+\delta^i_k\Gamma^a_{aj})-\frac{1}{n+1}(\delta^i_jS^a{}_{ka}+\delta^i_kS^a{}_{ja}).$$

In fact, this is equivalent to

$$\overline{\left\{{i \atop jk}\right\}}-\frac{1}{n+1}\left(\delta^j_i\overline{\left\{{a \atop ak}\right\}}+\delta^i_k\overline{\left\{{a \atop aj}\right\}}\right)=\left\{{i \atop jk}\right\}-\frac{1}{n+1}\left(\delta^i_j\left\{{a \atop ak}\right\}+\delta^i_k\left\{{a \atop aj}\right\}\right).$$

but, we take the above form. Then if we put

$$\Pi^i_{jk}=\Gamma^i_{jk}-\frac{1}{n+1}(\delta^i_j\Gamma^a_{ak}+\delta^i_k\Gamma^a_{aj}) \tag{3.14}$$

$$\Lambda^i_{jk}=S^i{}_{jk}+S^i{}_{kj}-\frac{1}{n+1}(\delta^i_jS^a{}_{ka}+\delta^i_kS^a{}_{ja}),$$

(3, 13) is reduced to

$$\overline{\Pi}^i_{jk}+\overline{\Lambda}^i_{jk}=\Pi^i_{jk}+\Lambda^i_{jk},$$

Here, Π^i_{jk}, so called T. Y. Thomas projective connection [4], is invariant for the projective transformations of the paths on the metric manifold with torsion; we have the theorem:

THEOREM *On the metric manifolds with torsion, if the tensor Λ^i_{jk} be invariant for the projective transformation on the Riemannian manifold with the same metric, then the metric manifolds are projectively transformable, and conversely.*

Next, we assume that the metric manifold with torsion are projectively transformable, i. e.,

$$\overline{\Pi}^i_{jk}=\Pi^i_{jk} \tag{3.15}$$

The equation of coordinate transformation is

$$\Pi'^i_{jk}=\frac{\partial x'^i}{\partial x^a}\left(\frac{\partial^2x^a}{\partial x'^j\partial x'^k}+\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\Pi^a_{bc}\right)$$

$$-\frac{1}{n+1}\left(\delta^i_j\frac{\partial\log\Delta}{\partial x'^i}-\delta^i_k\frac{\partial\log\Delta}{\partial x'^j}\right),$$

where $\Delta=\left|\frac{\partial x'}{\partial x}\right|$. From theorem in §2, we can aware that Π^i_{jk} is a connection in the metric manifold with torsion too, i. e.,

$$\frac{\partial x^a}{\partial x^{i'}}\Pi'^i_{jk}=\frac{\partial^2x^a}{\partial x'^j\partial x'^k}+\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}\Pi^a_{bc}. \tag{3.16}$$

Substituting the form obtained from (3.14)

$$\Gamma^i_{jk}=\Pi^i_{jk}+\frac{1}{n+1}(\delta^i_j\Gamma^a_{ak}+\delta^i_k\Gamma^a_{aj})$$

into (3.2), then the equation (3.2) of the paths is reduced to

$$\frac{d^2x^i}{ds^2}+\Pi^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}+\frac{2}{n+1}\Gamma^a_{ak}\frac{dx^k}{ds}\frac{dx^i}{ds}=0.$$

If we take the parameter p such that

$$(3,17)\qquad \frac{2}{n+1}\Gamma^a_{ak}\frac{dx^k}{ds}=\frac{\dfrac{d^2s}{dp^2}}{\left(\dfrac{ds}{dp}\right)^2},\quad \text{i.e.,}\quad p=\int e^{-\frac{2}{n+1}\int\Gamma^a_{ak}dx^k}ds,$$

then the equation of paths is reduced to

$$(3.18)\qquad \frac{d^2x^i}{dp^2}+\Pi^i_{jk}\frac{dx^j}{dp}\frac{dx^k}{dp}=0,$$

where p is invariant for the projective transformation, but not for the coordinate transformation [4]. In fact, in the coordinate system, we put the equation of the paths

$$\frac{d^2x^i}{dp'^2}+\Pi'^i_{jk}\frac{dx'^j}{dp'}\frac{dx'^k}{dp'}=0.$$

Substituting (3.16) into this and comparing with (3.18), then we obtain the relation

$$(3.19)\qquad p'=cp+m.$$

where c and m are arbitrary constants.

§ 4. The conformal Transformations

For two metric manifolds with torsion of which metrics are g_{ij} and $\bar{g}_{ij}$ we call that they are conformal, if there is the relation

$$(4.\ 1)\qquad \bar{g}_{ij}=e^{2\sigma}g_{ij}.$$

In this section, we will investigate about the relations of conformality between the Riemannian manifold and the metric manifold with torsion.

From (1.9), if we put

$$(4.\ 2)\qquad L_{jkl}{}^i=L_{jk}{}^i{}_{,l}-L_{jl}{}^i{}_{,k}+L_{jk}{}^sL_{sl}{}^i-L_{jl}{}^sL_{sk}{}^i,$$

then (1.9) is reduced to

$$(4.\ 3)\qquad E^i_{jkl}=R^i_{jkl}+L_{jkl}{}^i.$$

We well know that when the metric manifolds are conformally transformable, the conformal curvature tensor C^i_{jkl} is invariant [3], where

$$(4.\ 4)\qquad C^h_{ijk}=R^h_{ijk}+\frac{1}{n-2}[\delta^h_j R_{ik}-\delta^h_k R_{ij}+g_{ik}R^h_l-g_{ij}R^h_k]$$

$$+\frac{R}{(n-1)(n-2)}(g_{ij}\delta^h_k-g_{ik}\delta^h_j),$$

From (1.7) and (1.8), substituting (4.3) into (4.4), we obtain the form

$$(4.\ 5)\qquad C^h_{ijk}=C(E)^h_{ijk}-C(L)^h_{ijk},$$

where

$$(4.\ 6)\qquad C(E)^h_{ijk}=E^h_{ijk}+\frac{1}{n-2}(\delta^h_l E_{ik}-\delta^h_k E_{ij}+g_{ik}E^h_l-g_{ij}E^h_k)$$

$$+\frac{E}{(n-1)(n-2)}(g_{ij}\delta^h_k-g_{ik}\delta^h_j),$$

$$C(L)^h_{ijk}=L_{ijk}{}^h+\frac{1}{n-2}(\delta^h_j L_{ik}-\delta^h_k L_{ij}+g_{ik}L^h_j-g_{ij}L^h_k)$$

$$+\frac{L}{(n-1)(n-2)}(g_{ij}\delta^h_k-g_{ik}\delta^h_l),$$

$$E_{ij}=E^h_{ijh},\quad E=g^{ij}E_{ij;},\quad L_{ij}=L_{ijh}{}^h\quad L=g^{ij}L_{ij},$$

and consequently, for the metric manifolds with torsion the conformal invariant tensor is represened by (4.5). Since, when $C^h_{ijk}=0$, the metric manifold be conformally flat [3], then we get the theorem:

THEOREM *The necessary and sufficient condition that the metric manifold with torsion be conformally flat is that*

$$(4.\ 7)\qquad C(E)^h_{ijk}=C(L)^h_{ijk}$$

Next, differentiating (4.1) with respect to x^k we get

$$\frac{\partial\bar{g}_{ij}}{\partial x^k}=e^{2\sigma}\Big(\frac{\partial g_{ij}}{\partial x^k}+2\sigma_{,k}g_{ij}\Big),$$

and because of (1.1), this is reduced to

$$\bar{g}_{sj}\bar{L}_{ik}{}^s+\bar{g}_{is}\bar{L}_{jk}{}^s=e^{2\sigma}(g_{sj}E^s_{ik}+g_{is}E^s_{ik}+2\sigma_{,k}g_{ij}).$$

Since from (4.1), we get

$$\bar{g}^{jl}=e^{-2\sigma}g^{jl}.$$

Multiplying above form by this and summing for j, we obtain

$$E^i_{jk}+\bar{g}^{in}\bar{g}_{jm}\bar{E}^m_{nk}=E^i_{jk}+g^{in}g_{jm}E^m_{nk}+2\delta^i_j\sigma,_k, \tag{4.8}$$

and contracting for i and j, we get

$$\sigma,_k=\frac{1}{n}(\bar{E}^a_{ak}-E^a_{ak}). \tag{4.9}$$

If we substitute this into (4.8), then we have the invariant quantities for the conformal transformation, i.e., they are

$$*C^i_{jk}=E^i_{jk}+g^{in}g_{jm}E^m_{nk}-\frac{2}{n}\delta^i_jE^a_{ak}, \tag{4.10}$$

and if we put C^i_{jk} the symmetric part of $*C^i_{jk}$ in j and k, i.e.,

$$C^i_{jk}=\Gamma^i_{jk}+\frac{1}{2}g^{in}(g_{im}E^m_{nk}+g_{km}E^m_{nj})-\frac{1}{n}(\delta^i_jE^a_{ak}+\delta^i_kE^a_{aj}), \tag{4.11}$$

then they are invariant too, but not the connection for the coordinate transformation and for the coordinate transformation they are taken by the equation

$$\frac{\partial x^a}{\partial x'^i}C'^i_{jk}=\frac{\partial^2 x^a}{\partial x'^j\partial x'^k}+\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}C^a_{bc} \tag{4.12}$$

$$+\frac{1}{2}g^{al}\left(g_{bs}\frac{\partial x^b}{\partial x'^j}\frac{\partial x'^n}{\partial x^l}\frac{\partial^2 x^s}{\partial x'^n\partial x'^k}+g_{cs}\frac{\partial x}{\partial x'^k}\frac{\partial x'^n}{\partial x^l}\frac{\partial^2 x^s}{\partial x'^n\partial x'^j}\right).$$

If we substitute (4.11) in the equation (3.2) of paths, then the equation is reduced to

$$\frac{d^2x^i}{ds^2}+C^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}+\left(\frac{2}{n}E^a_{ak}\frac{dx^k}{ds}-\phi\right)\frac{dx^i}{ds}=0,$$

where we have taken ϕ to that

$$\phi\frac{dx^i}{ds}=g^{in}g_{jm}E^m_{nk}\frac{dx^j}{ds}\frac{dx^k}{ds} \tag{4.13}$$

everywhere on the paths. Taking the parameter p such that

$$\frac{2}{n}E^a_{ak}\frac{dx^k}{ds}-\phi=\frac{\frac{d^2s}{dp^2}}{\left(\frac{ds}{dp}\right)^2},\quad p=\int e^{-\int\left(\frac{2}{n}E^a_{ak}\frac{dx^k}{ds}-\phi\right)ds}ds, \tag{4.14}$$

then the equation of the paths is changed to

$$(4.15)\qquad \frac{d^2x^i}{dp^2}+C^i_{jk}\frac{dx^j}{dp}\,\frac{dx^k}{dp}=0.$$

Hence we have the following theorem:

THEOREM. *For the conformal transformation, the quantities* C_{ijk} *given by* (4.11) *are invariant in the metric manifold with torsion, and if we take the parameter* p *given by* (4.14), *the equation of the paths is changed to* (4.15) *and the parameter and the equation of the paths* (4.15) *are invariant for the conformal transformations.*

This parameter p is invariant for the conformal transformations. We will seek the relation of p for the coordinate transformations.

In (4.12), if we put

$$(4.16)\qquad \theta^i_{sk}=\frac{1}{2}g'^{im}g_{st}\frac{\partial^2 x^t}{\partial x'^m\partial x'^k},$$

then (4.12) is reduced to

$$(4.17)\qquad C'^i_{jk}=\frac{\partial x'^i}{\partial x^a}\left(\frac{\partial^2 x^a}{\partial x'^j\partial x'^k}+\frac{\partial x^b}{\partial x'^j}\,\frac{\partial x^c}{\partial x'^k}C^a_{bc}\right)+\frac{\partial x^s}{\partial x'^j}\theta^i_{sk}+\frac{\partial x^s}{\partial x'^k}\theta^i_{sj}.$$

In the coordinate system (x'), we put the equation of the paths

$$\frac{d^2x'^i}{dp'^2}+C'^i_{jk}\frac{dx'^j}{dp'}\,\frac{dx'^k}{dp'}=0.$$

Substituting (4.17) into this and making use of (4.15), we obtain the result

$$2\left(\frac{dp}{dp'}\right)^2\theta'^a_{sc}\,\frac{\partial x'^s}{\partial x^b}\,\frac{dx^b}{dp}\,\frac{dx^c}{dp}=\frac{dx^a}{dp}\,\frac{d^2p}{dp'^2},$$

where we put similarly to (4.16)

$$(4.18)\qquad \theta'^a_{sc}=\frac{1}{2}g^{at}g'_{sk}\frac{\partial^2 x'^k}{\partial x^t\partial x^c}.$$

And putting

$$(4.19)\qquad \tau^a=\frac{\theta'^a_{sc}\dfrac{\partial x'^s}{\partial x^b}\,\dfrac{dx^b}{dp}\,\dfrac{dx^c}{dp}}{\dfrac{dx^a}{dp}},$$

then we get the differential equation

$$\frac{d^2p}{dp'^2}=2\tau^a\left(\frac{dp}{dp'}\right)^2,$$

and solving it

$$p'=\int e^{-\int\tau_a dp}\,dp. \tag{4.20}$$

Next, we seek the condition of the integrability of (4.17).

Putting

$$u_i^a=\frac{\partial x^a}{\partial x'^i}, \tag{4.21}$$

then (4.17) is that

$$\frac{\partial u_j^a}{\partial x'^k}=u_l^a C'^i{}_{jk}-u_j^b u_k^c C_{bc}^a-u_j^a u_j^s\theta_{sk}^j-u_j^a u_k^s\theta_{sj}^i.$$

Differentiating with repect to x'^l, interchanging k and l, subtracting these equations and making usc of the condition of the integrability $\frac{\partial^2 u_j^a}{\partial x'^k\partial x'^l}=\frac{\partial^2 u_j^a}{\partial x'^l\partial x'^k}$, then we obtain

$$\begin{aligned}u_i^{a*}C'^i{}_{jkl}={}&u_j^b u_k^c u_l^{d*}C^a{}_{bcd}+u_l^a u_j^s\left(\frac{\partial\theta_{sl}^i}{\partial x'^k}-\frac{\partial\theta_{sk}}{\partial x'^l}+C'^i{}_{mk}\theta_{sl}^m-C'^j{}_{ml}\theta_{sk}^m\right)\\&-u_i^a u_k^s\left(\frac{\partial\theta_{sj}^i}{\partial x'^l}+C'^i{}_{ml}\theta_{sj}^m-C'^m_{jl}\theta_{sm}^i\right)+u_i^a u_l^s\left(\frac{\partial\theta_{sj}^i}{\partial x'^k}+C'^i{}_{mk}\theta_{sj}^m-C'^m_{jk}\theta_{sm}^i\right)\\&-u_i^a u_j^s u_k^t(C_{st}^r\theta_{rl}^i+\theta_{sl}^m\theta_{tm}^i)+u_i^a u_k^s u_l^t(\theta_s^m\theta_{tm}^i-\theta_{tj}^m\theta_{sm}^i),\\&+u_i^a u_l^s u_j^t(C_{st}^r\theta_{rk}^i+\theta_{sm}^i\theta_{tk}^m),\end{aligned}$$

where

$$*C_{jkl}^i=\frac{\partial}{\partial x^l}C_{jk}^i-\frac{\partial}{\partial x^k}C_{jl}^i+C_{jk}^m C_{ml}^i-C_{jl}^m C_{mk}^i.$$

Multiplying this by $v_a^l=\frac{\partial x'^l}{\partial x^a}$ and contracting, it is reduced to

$$*C'^l{}_{jkl}=u_j^b u_k^c\ {*C_{bca}^a}+u_j^s\left(\frac{\partial\theta_{sl}^l}{\partial x'^k}-\frac{\partial\theta_{sk}^l}{\partial x'^l}+C'^l{}_{mk}\theta_{sj}^m-C'^m_{lk}\theta_{sm}^l\right)$$

$$-u_k^s\left(\frac{\partial\theta^l_{sj}}{\partial x'^l}+C'^l{}_{ml}\theta^m_{sj}-C'^m{}_{jl}\theta^l_{sm}\right)+u_l^s\left(\frac{\partial\theta^l_{sj}}{\partial x'^k}+C'^l{}_{mk}\theta^m_{sj}-C'^m{}_{jk}\theta^l_{sm}\right)$$

$$-u_j^s u_k^t(C^r_{st}\theta^l_{rl}+\theta^m_{sl}\theta^l_{tm})+u_k^s u_l^t(\theta^m_{sj}\theta^l_{tm}-\theta^m_{tj}\theta^l_{sm})+u_l^s u_j^t(C^r_{st}\theta^l_{rk}+\theta^l_{sm}\theta^m_{tk}),$$

and putting

(4.22) $$*C'_{[jk]}={*C}^l{}_{jkl}-{*C}^l{}_{kjl}.$$

then we have the condition of the integrability, i.e.

(4.23) $$*C'_{[jk]}=u_j^b u_k^c{*C}_{[bc]}+u_j^s\left[\frac{\partial\theta^l_{sl}}{\partial x'_k}+(C^r_{st}\theta^l_{rk}+\theta^l_{sm}\theta^m_{tk})u_l^t\right]$$

$$-u_k^s\left[\frac{\partial\theta^l_{sl}}{\partial x'^j}+(C^r_{st}\theta^l_{rj}+\theta^l_{sm}\theta^m_{tj})u_l^t\right]$$

$$+u_l^s\left[\frac{\partial\theta^l_{sj}}{\partial x'^k}-\frac{\partial\theta^l_{sk}}{\partial x'^j}+C'^l{}_{mk}\theta^m_{sj}-C'^l{}_{mj}\theta^m_{sk}\right].$$

Hence we have the theorem:

THEOREM. *In the metric manifold with torsion, a necessary and sufficient condition that two systems of functions* $C^a_{bc}(x)$ *and* $C'^i{}_{jk}(x')$ *in the coordinate systems* (x) *and* (x') *respectively give the same systems of paths, is that they satisfy the condition* (4.23).

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REFERENCES

[1] K. Yano, S. Bochner. *Curvature and Betti numbers.* Annalt of Math. 1953
[2] L.P. *Eisenhart. Non-Riemannian geometry.* Amer. Math. Soc. Coll. Publ. VIII. 1926.
[3] L.P. Eisenhart. *Riemannian geometry.* Princeton University Press. 1926.
[4] K. Yano. *Geometry of paths,* Kyoritsu. Japan. 1949.