NOTES ON KAEHLERIAN METRIC

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§ 1. Introduction

In the complex n-dimensional (real $2n$-dimensional) analytic space in the sense of S. Bochner[1] (*), the coordinates of a point P may be considered as $(z^\alpha, \bar{z}^\alpha)$, where Greek indices take the values $1, 2, \cdots, n$.

If we put

$$\bar{z}^\alpha = z^{\bar{\alpha}} = conj.\ of\ z^\alpha,$$

and assume that barred Greek indices take the values $\bar{1}, \bar{2}, \cdots, \bar{n}$, where $\bar{\alpha} = n + \alpha$.

Let us assume that there is given a positive definite quadratic differential form [3]

$$(1.\ 1) \qquad ds^2 = g_{jk} dz^j dz^k, \qquad (j, k = 1, 2, \cdots, 2n)$$

where the symmetric tensor g_{jk} is self-adjoint [3] and satisfies

$$(1.\ 2) \qquad g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0.$$

By virtue of condition (1, 2) the metric form (1. 1) can be written in the form

$$(1.\ 3) \qquad ds^2 = 2 g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta,$$

where

$$(1.\ 4) \qquad g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha} = \overline{g_{\bar{\alpha}\beta}} = \overline{g_{\beta\bar{\alpha}}},$$

and a metric (1.3) satisfying (1.4) is called a Hermitian metric[3].

Taking account of

$$g^{\alpha\beta} = g^{\bar{\alpha}\bar{\beta}} = 0, \qquad g^{\alpha\bar{\beta}} = g^{\bar{\beta}\alpha} = \overline{g^{\bar{\alpha}\beta}} = \overline{g^{\beta\bar{\alpha}}},$$

we obtain for the Christoffel symbols Γ^i_{jk} the relations

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\bar{\xi}} \left(\frac{\partial g_{\bar{\xi}\beta}}{\partial z^\gamma} + \frac{\partial g_{\bar{\xi}\gamma}}{\partial z^\beta} \right).$$

(*) Numbers in brackets refer to the references at the end of the paper.

(1. 5) $$\Gamma^{\alpha}_{\beta\bar{\gamma}}=\frac{1}{2}g^{\alpha\bar{\varepsilon}}\left(\frac{\partial g_{\beta\bar{\varepsilon}}}{\partial \bar{z}^{\gamma}}-\frac{\partial g_{\beta\bar{\gamma}}}{\partial \bar{z}^{\varepsilon}}\right),$$

$$\Gamma^{\alpha}_{\bar{\beta}\bar{\gamma}}=0.$$

The condition

(1, 6) $$\Gamma^{\alpha}_{\beta\bar{\gamma}}=0$$

is called Kaehlerian condition, and this is equivalent to

(1, 7) $$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^{\gamma}}=\frac{\partial g_{\gamma\bar{\beta}}}{\partial z^{\alpha}}.$$

or, further to

(1. 8) $$g_{\alpha\bar{\beta}}=\frac{\partial^2 \phi}{\partial z^{\alpha}\partial \bar{z}^{\beta}},$$

where ϕ be a real valued function.

A metric satisfying (1.4) and (1.7) will be called a Kaehlerian metric. Thus, in a Kaehlerian metric, we have

(1, 9) $$\Gamma^{\alpha}_{\beta\gamma}=g^{\alpha\bar{\varepsilon}}\frac{\partial g_{\beta\bar{\varepsilon}}}{\partial z^{\gamma}}, \qquad \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}=g^{\bar{\alpha}\varepsilon}\frac{\partial g_{\varepsilon\bar{\beta}}}{\partial \bar{z}^{\gamma}},$$

(1.10) $$R^{\alpha}_{\beta\gamma\bar{\delta}}=\frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial \bar{z}^{\delta}},$$

where R^i_{jkl} was defined

$$R^i_{jkl}=\frac{\partial \Gamma^i_{jk}}{\partial z^l}-\frac{\partial \Gamma^i_{jl}}{\partial z^k}+\Gamma^s_{jk}\Gamma^i_{sl}-\Gamma^s_{jl}\Gamma^i_{sk}.$$

In the present paper, we study on the Kaehlerian metric.

At first, in section 2, we introduce orthogonal ennuple which is constructed by T. Suguri in the [2], and we find the equivalent two conditions to the Kaehlerian condition for the complex analytic ennuple, and a sufficient condition to the Kaehlerian condition for the non-complex analytic ennuple. Next, in section 3, we determine the constant holomorphic curvature space in the general Fubini space, and in section 4, we have a necessary and sufficient condition that the Kaehlerian space is a space of constant holomorphic curvature.

And in this paper we always assume the self-adjointness on the indices.

§ 2. Orthogonal Ennuple

We can introduce the mutually orthogonal vectors $\eta_{p|}{}^{\alpha}$ such as

$$(2.\ 1)\qquad g_{\alpha\beta}\eta_{p|}{}^{\alpha}\overline{\eta_{q|}{}^{\beta}}=\delta_{pq},\qquad |\eta_{p|}{}^{\alpha}|\neq 0,\qquad (p, q=1, \cdots, n),$$

in the Hermitian space (see[2]).

From the self-adjointness of $\eta_{p|}{}^{\alpha}$,

$$(2.\ 2)\qquad \overline{\eta_{q|}{}^{\beta}}=\eta_{q|}{}^{\bar{\beta}}=conj.\ of\ \eta_{q|}{}^{\beta},$$

then

$$(2.\ 3)\qquad g_{\alpha\bar{\beta}}\eta_{p|}{}^{\alpha}\eta_{q|}{}^{\bar{\beta}}=\delta_{pq},\qquad (p, q=1, \cdots, n).$$

If we solve the equations

$$(2.\ 4)\qquad g_{\alpha\bar{\beta}}\eta_{p|}{}^{\alpha}=\eta_{p|\bar{\beta}},\qquad g_{\alpha\bar{\beta}}\eta_{p|}{}^{\bar{\beta}}=\eta_{p|\alpha},$$

then

$$(2.\ 5)\qquad g_{\alpha\bar{\beta}}=\sum_{p}\eta_{p|\alpha}\eta_{p|\bar{\beta}},\qquad \sum_{p}\eta_{p|\alpha}\eta_{q|}{}^{\gamma}=\delta^{\gamma}_{\alpha},\qquad g^{\alpha\bar{\beta}}=\sum_{p}\eta_{p|}{}^{\alpha}\eta_{p|}{}^{\bar{\beta}}.$$

Here we envisage the following conditions in the Hermitian space.

(α) $\eta_{p|}{}^{\alpha}$, $\eta_{p|\alpha}(conj.)$ are complex analytic in z^{α} $(conj.)$

(A) $\eta_{p|}{}^{\alpha}$, $\eta_{p|\alpha}$ are parallel in the constructed metric.

(B) Kaehlerian condition.

(C) A transitive group of transformations whose infinitesimal operators are

$$X_{p}f=\eta_{p|}{}^{\alpha}\frac{\partial f}{\partial z^{\alpha}}$$

is commutative.

(I) At first we assune the condition (α).

If the Kaehlerian condition (B) is satisfied then, by the covariant differentiation, we get

$$(2.\ 6)\qquad \eta_{p|}{}^{\gamma}{}_{;\beta}=g^{\bar{\alpha}\gamma}\eta_{p|\bar{\alpha};\beta}=g^{\bar{\alpha}\gamma}\frac{\partial\eta_{p|\bar{\alpha}}}{\partial z^{\beta}},$$

$$\eta_{p|\gamma;\beta}=g_{\gamma\bar{\alpha}}\eta_{p|}{}^{\bar{\alpha}}{}_{;\beta}=g_{\gamma\bar{\alpha}}\frac{\partial\eta_{p|}{}^{\bar{\alpha}}}{\partial z^{\beta}},$$

where semicolons indicate the covariant derivatives,

Then we can easily see the following:

LEMMA *If an orthogonal eunuple* (2. 1) *satisfies (B), the conditions* (α) *and* (A) *are equlvalent.*

Under the assumption (α), the Kaehlerian condition is equivalent to the curl property

$$\frac{\partial\eta_{p|\alpha}}{\partial z^{\beta}}=\frac{\partial\eta_{p|\beta}}{\partial z^{\alpha}}. \tag{2. 7}$$

(i) If (A) is satisfied. then we have

$$\eta_{p|\alpha;\gamma}-\eta_{p|\gamma;\alpha}=0, \tag{2. 8}$$

and this is equivalent to the curl property (2.7), thus the condition (B) is satisfied, then

$$\eta_{p|}{}^{\alpha}{}_{;\gamma}=\frac{\partial\eta_{p|}{}^{\alpha}}{\partial z^{\gamma}}+\eta_{p|}{}^{\beta}\Gamma^{\alpha}_{\beta\gamma}=0.$$

From (1. 9) and (2. 5) we get

$$\frac{\partial\eta_{p|}{}^{\alpha}}{\partial z^{\gamma}}-\eta_{p|}{}^{\beta}\sum_{t}\eta_{t|\gamma}\frac{\partial\eta_{t|}{}^{\alpha}}{\partial z^{\beta}}=0.$$

Multiplying $\eta_{q|}{}^{\gamma}$ and summing up with respect to γ, we find

$$\eta_{q|}{}^{\beta}\frac{\partial\eta_{p|}{}^{\alpha}}{\partial z^{\beta}}-\eta_{p|}{}^{\beta}\frac{\eta_{q|}{}^{\alpha}}{\partial z^{\beta}}=0,$$

thus the condition (C) is satisfied.

(ii) If (B) is satisfied, multiplying (1.7) by $g^{\alpha\bar{\sigma}}g^{\gamma\bar{\rho}}$ and snmming up with respect to α and γ, we find

$$g^{\alpha\bar{\sigma}}g^{\gamma\bar{\rho}}\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^{\gamma}}=g^{\alpha\bar{\sigma}}g^{\gamma\bar{\rho}}\frac{\partial g_{\gamma\bar{\beta}}}{\partial z^{\alpha}}. \tag{2. 9}$$

By using of the relation

$$g_{\alpha\beta}g^{\bar{\gamma}\alpha}=\delta^{\bar{\gamma}}_{\bar{\beta}}$$

to (2.9) and multiplying the result by $g^{\beta\lambda}$ and summing up with respect to $\bar{\beta}$, we have the condition

$$g^{\gamma\bar{\rho}}\frac{\partial g^{\alpha\bar{\sigma}}}{\partial z^{\gamma}}=g^{\gamma\bar{\sigma}}\frac{\partial g^{\alpha\bar{\rho}}}{\partial z^{\gamma}}. \tag{2.10}$$

which is equivalent to (1.7).

By substituting (2.5) into this condition, we have

$$\sum_{p,q}\left(\eta_{q|}{}^{\gamma}\eta_{q|}{}^{\bar{\rho}}\eta_{p|}{}^{\bar{\sigma}}\frac{\partial\eta_{p|}{}^{\alpha}}{\partial z^{\gamma}}\right)=\sum_{p,q}\left(\eta_{q|}{}^{\gamma}\eta_{q|}{}^{\bar{\sigma}}\eta_{p|}{}^{\bar{\rho}}\frac{\partial\eta_{p|}{}^{\alpha}}{\partial z^{\gamma}}\right).$$

Multiplying $\eta_{t|\bar{\rho}}\eta_{s|\bar{\sigma}}$ and summing up with respect to $\bar{\rho}$ and $\bar{\sigma}$, we find

$$\eta_{t|}{}^{\gamma}\frac{\partial\eta_{s|}{}^{\alpha}}{\partial z^{\gamma}}=\eta_{s|}{}^{\gamma}\frac{\partial\eta_{t|}{}^{\alpha}}{\partial z^{\gamma}},$$

thus the condition (C) is satisfied

In this case by the lemma we see that the condition (A) also is satisfied

(iii) If (C) is satisfied, then, by puting

$$g^{\alpha\beta}=\sum_{p}\eta_{p|}{}^{\alpha}\eta_{p|}{}^{\beta}$$

instead of (2.5), K. Yano and S. Bochner have proved that the conditions (A) and (B) are satisfied ([3] pp. 134-135).

Hence we have the following

THEOREM 1. *If an orthogonal ennuple* (2.1) *is complex analytic in the Hermitian space, the conditions (A), (B) and (C) are equivalent.*

If above complex analytic ennuple satisfies one among the conditions (A), (B) and (C), then from the Ricci identity:

$$\eta_{p|}{}^{\alpha}{}_{;\gamma;\delta}-\eta_{p|}{}^{\alpha}{}_{;\delta;\gamma}=\eta_{p|}{}^{\beta}R^{\alpha}{}_{\beta\gamma\delta},$$

we get easily

$$R^{\alpha}{}_{\beta\gamma\bar{\delta}}=0,$$

thus, our space is a flat Kaehlerian space [3].

For the infinitesimal point transformation

$$'z^{\alpha}=z^{\alpha}+\eta_{p|}{}^{\alpha}(z)\delta t, \tag{2.11}$$

the Lie derivatives of $g_{\alpha\beta}$ and $\Gamma^{\alpha}_{\beta\gamma}$ are given as follows

$$\pounds g_{\alpha\beta}=\eta_{p|\alpha;\beta}+\eta_{p|;\beta\alpha},$$

$$\pounds\Gamma^{\alpha}_{\beta\gamma}=\eta_{p|}{}^{\alpha}{}_{;\beta;\gamma}+K^{\alpha}_{\beta\gamma\delta}\eta_{p|}{}^{\delta}$$

Then we have the following:

COROLLARY *If an orthogonal ennuple* (2.1) *is complex analytic and satisfies (B), then the infinitesimal point transformation* (2.11) *is a motion and an infinitesimal affine collineation in the Kaehlerian space.*

(II) We assume that the ennuple is not complx analytic, and in this case we put (C′), instead of above (C), and the following (X).

(X) $\qquad \eta_{t|\bar{\rho}}\eta_{s|}{}^{\beta}=\eta_{s|\bar{\rho}}\eta_{t|}{}^{\beta}.$

(B) $\qquad$ Kaehlerian condition,

(C′) $\qquad \eta_{p|}{}^{\beta}\dfrac{\partial\eta_{q|}{}^{\alpha}}{\partial z^{\beta}}=\eta_{q|}{}^{\beta}\dfrac{\partial\eta_{p|}{}^{\alpha}}{\partial z^{\beta}}.$

Here we assume the condition (X), then multiplying (X) by $\dfrac{\partial g^{\alpha\bar{\rho}}}{\partial z^{\beta}}$ and summing up with respect to β and $\bar{\rho}$, we get, by (2.4)

$$\eta_{s|}{}^{\beta}\Big(\frac{\partial\eta_{t|}{}^{\alpha}}{\partial z^{\beta}}-g^{\alpha\bar{\rho}}\frac{\partial\eta_{t|\bar{\rho}}}{\partial z^{\beta}}\Big)=\eta_{t|}{}^{\beta}\Big(\frac{\partial\eta_{s|}{}^{\alpha}}{\partial z^{\beta}}-g^{\alpha\bar{\rho}}\frac{\partial\eta_{s|\bar{\rho}}}{\partial z^{\beta}}\Big), \tag{2.12}$$

and into this result we substitute (2.5), and multiplying this by $\eta_{s|}{}^{\bar{\gamma}}\eta_{t|}{}^{\bar{\lambda}}$ and summing up with respect to s and t, we have

$$g^{\beta\bar{\gamma}}\frac{\partial g^{\alpha\bar{\lambda}}}{\partial z^{\beta}}=g^{\beta\bar{\lambda}}\frac{\partial g^{\alpha\bar{\gamma}}}{\partial z^{\beta}},$$

thus, the Kaehlerian condition is satisfied. (2.10)

On the other hand, if we differentiate (X) with respect to z^{γ}, and multiplying this derivative by $g^{\alpha\bar{\rho}}$ and summing up with respect to $\bar{\rho}$, and contracting by $\alpha=\gamma$. we get

(2.13) $$\eta_{t|}{}^{\alpha}\frac{\partial\eta_{s|}{}^{\beta}}{\partial z^{\alpha}}+g^{\alpha\bar{\rho}}\eta_{s|}{}^{\beta}\frac{\partial\eta_{t|\bar{\rho}}}{\partial z^{\alpha}}=\eta_{s|}{}^{\alpha}\frac{\partial\eta_{t|}{}^{\beta}}{\partial z^{\alpha}}+g^{\alpha\bar{\rho}}\eta_{t|}{}^{\beta}\frac{\partial\eta_{s|\bar{\rho}}}{\partial z^{\alpha}}.$$

From this by using the condition (X), we find

$$\eta_{t|}{}^{\alpha}\frac{\partial\eta_{s|}{}^{\beta}}{\partial z^{\alpha}}=\eta_{s|}{}^{\alpha}\frac{\partial\eta_{t|}{}^{\beta}}{\partial z^{\alpha}},$$

thus the condition (C′) is satisfied.

For the complex analytic ennuple, we can easily see from (2.13) and theorem 1 that if (X) is assumed then (B) and (C′) are satisfied.

Hence we have

THEOREM 2. *If the condition (X) is satisfied for the above orthogonal ennuple (complex analytic or not), then the conditions (B) and (C′) are satisfied.*

§ 3. General Fubini space

For arbitrary real b, S. Bochner [1] put

(3. 1) $$\phi=\frac{2}{b}\log\left(1+\frac{b}{2}\sum_{\alpha=1}^{n}z_{\alpha}\bar{z}_{\alpha}\right)\equiv\frac{2}{b}\log S,$$

then

(3. 2) $$g_{\alpha\beta}=\frac{\partial^2\phi}{\partial z^{\alpha}\partial\bar{z}^{\beta}}=\frac{1}{S}\delta_{\alpha\beta}-\frac{b}{2S^2}\bar{z}_{\alpha}z_{\beta},$$

thus a Kaehlerian metric is constructed.

From this we can calculate as follows

(3. 3) $$\Gamma^{\alpha}_{\beta\gamma}=g^{\alpha\bar{\xi}}\frac{\partial g_{\bar{\xi}\beta}}{\partial z^{\gamma}}=-\frac{b}{2S}(\bar{z}_{\beta}\delta^{\alpha}_{\gamma}+\bar{z}_{\gamma}\delta^{\alpha}_{\beta}),$$

(3. 4) $$\Gamma^{\alpha}_{\beta\gamma}=-\frac{b(n+1)}{2S}\bar{z}_{\beta},$$

(3. 5) $$R_{\beta\bar{\gamma}}=\frac{b}{2}(n+1)g_{\beta\bar{\gamma}},$$

the last result is the theorem 8 in [1].

From (3.3) we get

(3. 6) $$R^{\alpha}{}_{\beta\gamma\bar{\delta}}=-\frac{b}{2}(\delta^{\alpha}_{\beta}g_{\gamma\bar{\delta}}+\delta^{\alpha}_{\gamma}g_{\beta\bar{\delta}}),$$

$$R_{\alpha\beta\gamma\delta}=\frac{2}{b}(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\delta}g_{\gamma\beta}). \tag{3. 7}$$

By using the expression of [3] we have

$$\frac{1}{2}R=g^{\beta\bar{\gamma}}R_{\beta\bar{\gamma}}=\frac{b}{2}n(n+1), \tag{3. 8}$$

and, by using the above

$$R_{\beta\bar{\gamma}}=\frac{R}{2n}g_{\beta\bar{\gamma}}, \tag{3. 9}$$

$$R_{\alpha\beta\gamma\delta}=\frac{R}{2n(n+1)}(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\delta}g_{\gamma\beta}), \tag{3. 10}$$

thus, the Fubini space is a space of constant holomorphic curvature [1], [3].

Furthermore, from (3. 9) ([3] pp. 131)

$$R_{;\alpha}=0,$$

then

$$R_{\alpha\beta\gamma\delta;\lambda}=0,$$

thus, the Fubini space is a type of symmetric space.

More generally we put

$$\phi=f(p), \qquad p=\sum_{\alpha=1}^{n}z_{\alpha}\bar{z}_{\alpha}, \tag{3. 11}$$

and

$$g_{\alpha\beta}=\frac{\partial^2\phi}{\partial z^{\alpha}\partial\bar{z}^{\beta}}=f'(p)\delta_{\alpha\beta}+f''(p)\bar{z}_{\alpha}z_{\beta}, \tag{3. 12}$$

then from this we can calculate

$$\Gamma^{\alpha}_{\beta\gamma}=\frac{f''}{f'}[\bar{z}_{\beta}\delta^{\alpha}_{\gamma}+\bar{z}_{\gamma}\delta^{\alpha}_{\beta}]+\varphi(p)z_{\alpha}\bar{z}_{\beta}\bar{z}_{\gamma}, \tag{3. 13}$$

where dashes mean the derivatives with respect to p and

$$\varphi(p)=\frac{f'(p)f'''(p)-2(f''(p))^2}{f'(p)[f'(p)+pf''(p)]}, \tag{3. 14}$$

then we have also

$$R_{\alpha\beta\gamma\delta}=-f''[\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta}]$$

(3.15)
$$\frac{f'f''' - (f'')^2}{f'} z_\delta[\bar{z}_\alpha \delta_{\beta\gamma} + \bar{z}_\gamma \delta_{\alpha\beta}]$$
$$-\left[\varphi(f'+pf'') + \frac{(f'')^2}{f'}\right] z_\beta [\bar{z}_\alpha \delta_{\gamma\delta} + \bar{z}_\gamma \delta_{\alpha\delta}]$$
$$-\left[\varphi'(f'+pf'') + \frac{2f''(f'f''' - (f'')^2)}{(f')^2}\right] \bar{z}_\alpha z_\beta \bar{z}_\gamma z_\delta.$$

But on the other hand,

(3.16)
$$g_{\alpha\beta} g_{\gamma\delta} + g_{\alpha\delta} g_{\gamma\beta} = (f')^2 [\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}]$$
$$+ f'f''[z_\delta(\bar{z}_\delta \delta_{\beta\gamma} + \bar{z}_\gamma \delta_{\alpha\beta}) + z_\beta \bar{z}_\alpha \delta_{\gamma\delta} + \bar{z}_\gamma \delta_{\alpha\delta})]$$
$$+ 2(f'')^2 \bar{z}_\alpha z_\beta \bar{z}_\gamma z_\delta.$$

If above general Fubini space whose metric tensor is defined by (3.12) is a space of constant holomorphic curvature, then relation

(3.17)
$$R_{\alpha\beta\gamma\delta} = -b(g_{\alpha\beta} g_{\gamma\delta} + g_{\alpha\delta} g_{\gamma\beta})$$

must be satisfied at all points of the space.

Furthermore (3.17) must be satisfied for all indices.

We consider for the point at least one of whose coordinates, say z^λ ($1 \leqq \lambda \leqq n$), is zero, in this case, by putting $\alpha = \beta = \gamma = \delta = \lambda$ we must hold

(3.18)
$$f'' = b(f')^2$$

at such point.

Therefore, we consider for the point all of whose coordinates are not zero.

At first, by considering the case all indices are distinct, we must hold

(3.19)
$$2b(f'')^2 = \varphi'(f'+pf'') + \frac{2f''(f'f''' - (f'')^2)}{(f')^2}$$

and equation (3.19) does not depend upon the indices, therefore we must hold (3, 19) at every such point of the space.

Next, by considering the case γ is equal to δ only, we must hold the following similarly from (3.17) and (3.19)

(3.20)
$$bf'f'' = \varphi(f'+pf'') + \frac{(f'')^2}{f'}.$$

Third, by considering the case α is equal to β only, we must hold the following similarly from (3.17) and (3.19)

(3.21) $$bf'f''=\frac{f'f'''-(f'')^2}{f'}.$$

Finally, from (3.17) and above three equations we must hold

(3.22) $$b(f')^2=f''.$$

Thus, we must hold above four equations simultaneously at every such point of the space.

But, (3.19), (3.20) and (3.21) are satisfied by (3.22).

Therefore the solution of above simultaneous equations is (3.22).

Therefore we can easily see from (3.18) and (3.22) that if (3.17) is satisfied, then we must hold the following at every point of the space

(3.23) $$f''(p)=b(f'(p))^2,$$

and this is equivalent to

(3.24) $$\varphi(p)=0.$$

Thus, if above general Fubini space is a space of constant holomorphic curvature, then we must hold equation (3.24).

Conversely, if the condition (3.24) holds for above space, then equations (3.17) also hold from (3.15) and (3.16).

Hence we get the following conclusion by solving the differential equation (3.24).

THEOREM 3. *General Fubini space whose metric is defined by* (3.11) *and* (3.12) *is a space of constant holomorphic curvature if and any if the function* (3.11) *is the form*

$$\phi=\frac{1}{a}\log(ap+b)+c$$

where $p=\sum_{\alpha=1}^{n}z_\alpha\bar{z}_\alpha$, $a(\neq 0)$, $b\ (\neq 0)$ *and* c *are integral constants.*

§4. Constant holomorphic curvature space

S. Bochner introduced the tensor $K_{\alpha\beta\gamma\bar{\delta}}$ such as

$$K_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}-\frac{1}{n+2}\left(g_{\alpha\beta}R_{\gamma\delta}+g_{\alpha\delta}R_{\gamma\beta}+g_{\gamma\delta}R_{\alpha\beta}+g_{\gamma\beta}R_{\alpha\delta}\right)$$

$$+\frac{R}{2(n+1)(n+2)}(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\delta}g_{\gamma\beta})$$

in the Kaehlerian space ([3] pp. 162).

For the tensor equation $K_{\alpha\bar{\beta}\gamma\bar{\delta}}=0$ is not the condition that the space is a space of constant holomorphic curvature, let us introduce

$$N_{\alpha\bar{\beta}\gamma\bar{\delta}}=R_{\alpha\bar{\beta}\gamma\bar{\delta}}-\frac{1}{2(n+2)}(g_{\alpha\bar{\beta}}R_{\gamma\bar{\delta}}+g_{\bar{\delta}\alpha}R_{\gamma\bar{\beta}}+g_{\gamma\bar{\delta}}R_{\alpha\bar{\beta}}+g_{\gamma\bar{\beta}}R_{\alpha\bar{\delta}}) - \frac{R}{2n(n+1)(n+2)}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}+g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}). \tag{4. 1}$$

If we assume $N_{\alpha\bar{\beta}\gamma\bar{\delta}}=0$ then by contracting with $g^{\alpha\bar{\delta}}$, we get

$$R_{\alpha\bar{\beta}}=\frac{R}{2n}g_{\alpha\bar{\beta}}, \tag{4. 2}$$

and substituting it in $N_{\alpha\bar{\beta}\gamma\bar{\delta}}=0$, we obtain

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}=\frac{R}{2n(n+1)}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}+g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}). \tag{4. 3}$$

Conversely, for the space of constant holomorphic curvature, we have (4.2) and (4.3), and hence evidently $N_{\alpha\bar{\beta}\gamma\bar{\delta}}=0$.

Hence we have the following conclusion.

THEOREM 4. *A necessary and sufficient condition that the Kaehlerian space is a space of constant holomorphic curvature, is that the tensor $N_{\alpha\bar{\beta}\gamma\bar{\delta}}$ vanishes.*

The theorem 8.24 in [3] as for Betti numbers can be utilized for the space satisfying $N_{\alpha\bar{\beta}\gamma\bar{\delta}}=0$.

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REFERENCES

[1] S. Bochner: *Curvature in Hermitian metric*, Bull. Amer. Math. Soc., Vol. 53, (1947).

[2] T. Suguri: *Theory of curves in the unitary K_n-connected space*, Mem. Fac. Sci. Kyūsyū Univ. (A). (6), (1951).

[3] K. Yano and S. Bochner: *Curvature and Betti numbers*, Ann. of Math. Studies No. 32, Princeton, (1953).