보형 공진기의 열탄성 감쇠 해석

Analysis of the Thermo-Elastic Damping of a Beam-Type Resonator

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ABSTRACT

This paper deals with the thermo-elastic damping (TED) due to the temperature change in a beam when it is in a resonant condition. Based on previous references, the analytical formulation for TED of a resonant thin beam was derived, and then TED was expressed as a function of the geometry of the beam, especially, its thickness. It was clearly shown that TED of a resonant beam is significantly varied for different thickness. Finally, the worst thickness of the beam has been identified in regard to the high-Q factor, and the result was compared to the finite element analysis.

1. Introduction

Thermo-elastic damping (TED) is related to the temperature change due to the expansion and contraction inside an elastic body when it is oscillating. The first study of TED was done by C.Zener(1). (See also L.D.Landau & E.M.Lifshitz(2).) Since then numerous researchers have performed TED related studies including mechanical & aerospace industries. TED for a nano-scale structure can also be found in R.Lifshitz, & M.L.Roukes(3).

TED has an important role in the design of a resonator to obtain high Q factor as possible. If the fused quartz is used as a material for such as the hemispherical resonator gyroscope the effect of TED may be secondary because other damping mechanisms are more governing. On the other hand, for the metals (like Elinvar™ and other steels) TED cannot be ignored.

This paper mainly deals with the study on the dependency of TED on the shape of a resonant thin beam. Especially, the worst thickness of the resonant beam in regard to the high Q factor is to be identified theoretically, so that the proposed procedure can be utilized for the design of a high Q beam resonator.

2. Thermo-Elastic Equation of a Resonant Beam

Let us consider the beam as shown in Fig.1. The beam is assumed to be resonating in a specific flexural mode. The boundary conditions at the left and right ends can be free, pinned, or fixed, respectively.

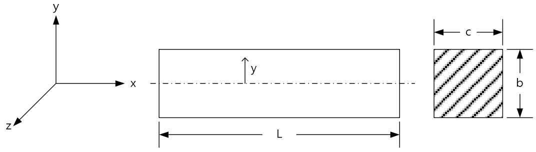

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Figure 1. Beam-type resonator

Considering the relationship between the stresses and strains including the effect of the heating for the thin resonating beam in Fig. 1, the force equilibrium equation[3] can be derived as Eq. (1).

$$\rho A\,\frac{\partial^2 Y}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 Y}{\partial x^2}+E\alpha I_T\right)=0\,, \qquad (1)$$

where ρ is the density, Y is the transverse displacement, $A=bc$ is the cross-section area, α is the linear thermal-expansion coefficient, and I is the moment of inertia of the cross section. I_T is defined as Eq. (2),

$$I_T=\int_A y\theta\, dydz, \qquad (2)$$

which is the thermal moment of inertia of the cross-section. In Eq. (2), θ is the relative temperature field. The rate of change of temperature is expressed as Eq. (3)[4].

$$\frac{\partial\theta}{\partial t}=\chi\,\nabla^2\theta-\frac{E\,\alpha\,T}{(1-2\sigma)C_v}\frac{\partial}{\partial t}\sum_j u_{jj} \quad , \qquad (3)$$

where $\chi=\kappa/C_p$ is the thermal diffusivity, κ is the thermal conductivity, and u_{jj} is the strain. The specific heat capacities (C_p, C_v) are volumetric ones. Eq. (3) can be expressed as Eq. (4) by considering the relationship between the strain, displacement and thermal expansion.

$$\left(1+2\frac{E\alpha^2 T_0}{C_p}\frac{1+\sigma}{1-2\sigma}\right)\frac{\partial\theta}{\partial t}=\chi\frac{\partial^2\theta}{\partial y^2}+y\frac{E\alpha\,T_0}{C_p}\frac{\partial}{\partial t}\frac{\partial^2 Y}{\partial x^2}\;, \qquad (4)$$

where T_0 is the equivalent temperature, and $E\alpha^2T_0/C_p$ $(=\Delta_E)$, is defined as the relaxation strength of Young's modulus.

By assuming $Y(x,t)=Y_0(x)e^{i\omega t}$, and $\theta(x,y,t)=\theta_0(x,y)e^{i\omega t}$, Eq. (4) can be expressed as Eq. (5).

$$\frac{\partial^2\theta_0}{\partial y^2}=i\frac{\omega}{\chi}\left(\theta_0-\frac{\Delta_E}{\alpha}\frac{\partial^2 Y_0}{\partial x^2}y\right). \qquad (5)$$

Then, by assuming the following relation,

$$\theta_0-\frac{\Delta_E}{\alpha}\frac{\partial^2 Y_0}{\partial x^2}y=A\sin(ky)+B\cos(ky), \quad (6)$$

and applying the insulation boundary condition at the upper and lower surfaces, Eq. (7) is obtained.

$$k=\sqrt{i\frac{\omega}{\chi}}=(1+i)\sqrt{\frac{\omega}{2\chi}} \quad , \qquad (7)$$

where k is a complex number. Then the equation for $\theta_0(x,y)$ is as follows:

$$\theta_0(x,y)=\frac{\Delta_E}{\alpha}\frac{\partial^2 Y_0(x)}{\partial x^2}\left(y-\frac{\sin(ky)}{k\cos\left(\frac{bk}{2}\right)}\right). \quad (8)$$

Now, by substituting Eq. (8) into Eq. (2), we obtain Eq. (9),

$$I_T=\frac{\Delta_E}{\alpha}\frac{\partial^2 Y_0(x)}{\partial x^2}\left[\frac{b^3c}{12}+\frac{2(k^2+1)c}{k^3}\tan\left(\frac{bk}{2}\right)\right], \quad (9)$$

and Eq. (1) becomes

$$\omega^2 Y_0=\frac{EI}{\rho A}\{1+\Delta_E[1+f(\omega)]\}\frac{\partial^4 Y_0}{\partial x^4}\;, \quad (10)$$

where $f(\omega)$ is

$$f(\omega)=f(k(\omega))=\frac{24}{b^3k^3}\left[\frac{bk}{2}-\tan\left(\frac{bk}{2}\right)\right]. \quad (11)$$

In Eq. (10), let us define $E\{1+\Delta_E[1+f(\omega)]\}=E_\omega$. The natural frequency equation of a beam[5] is

$$\omega_0=\beta^2\sqrt{\frac{EI}{\rho A}} \quad , \qquad (12)$$

where β is a coefficient determined by boundary conditions. Substituting E_ω into E (an isothermal Young's modulus) into Eq. (12), we get the complex natural frequency equation as Eq. (13).

$$\omega=\omega_0\left[1+\frac{\Delta_E}{2}[1+f(\omega_0)]\right] \quad . \qquad (13)$$

Therefore, we get

$$\mathrm{Re}(\omega)=\omega_0\left[1+\frac{\Delta_E}{2}\left(1-\frac{6}{\xi^3}\frac{\sinh\xi-\sin\xi}{\cosh\xi+\cos\xi}\right)\right], \quad (14)$$

$$\mathrm{Im}(\omega) = \omega_0 \frac{\Delta_E}{2}\left(\frac{6}{\xi^3}\frac{\sinh\xi + \sin\xi}{\cosh\xi + \cos\xi} - \frac{6}{\xi^2}\right), \quad (15)$$

where ξ is defined as

$$\xi = b\sqrt{\frac{\omega_0}{2\chi}} \quad . \quad (15)$$

TED can be calculated as the form of Q inverse as Eq. (16)[3]

$$Q^{-1} = 2\left|\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}\right|. \quad (16)$$

Using Eqs. (13), (14) and (16), we obtain the following TED equation:

$$Q^{-1} = \frac{E\alpha^2 T_0}{C_p}\left(\frac{6}{\xi^2} - \frac{6}{\xi^3}\frac{\sinh\xi + \sin\xi}{\cosh\xi + \cos\xi}\right) \ . (17)$$

Fig.2 shows TED value as a function of ξ. It is revealed that the maximum damping, which corresponds to the lowest Q-factor, occurs at ξ=2.225.

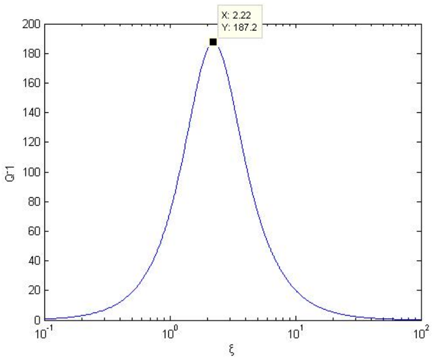

Figure 2. Thermo-elastic damping as a function of ξ

3. Prediction of the worst thickness corresponding to the lowest Q factor

Using Eqs. (12) and (15), the following relationship can be derived:

$$\xi^3 = b^3\beta^2\frac{1}{2\chi}\sqrt{\frac{E}{12\rho}} \quad . \quad (18)$$

The numerical values of βL are shown in Table 1 for different boundary conditions, therefore, β can be determined for a fixed L. In this paper L and c are fixed as 50 mm and 1 mm, respectively, as an example.

Table 1 βL for different boundary conditions

boundary condition	βL
free-free	4.73
fixed-pinned	3.92
fixed-free	1.87
pinned-pinned	3.14

Material properties of Aluminum in Table 2 are used for a numerical calculation.

Table 2 Material properties of Aluminum

Name	Value
mass density	2698.9 kg/m^3
Young's modulus	68 10^9 Pa
Poisson's ratio	0.36
thermal expansion coefficient	23.1 10^{-6} 1/K
heat capacity at constant pressure	900 J/kg K
thermal conductivity	210 W/m K

Using Eq. (18) and the maximum TED condition, ξ=2.225, as discussed in section 2, the worst thickness can be easily obtained, and the result is presented in Table 3

Table 3 Worst thickness for various boundary conditions in regard to high Q factor

Boundary condition	Worst thickness
fixed-fixed	0.404mm
fixed-pinned	0.458mm
fixed-free	0.75mm
pinned-pinned	0.53mm

4. Finite element analysis for TED and comparison with the analytical result

The finite element analysis (FEA) for TED evaluation was performed using COMSOL ver. 3.5. The typical model is shown in Fig. 3. The geometry of the beam and boundary conditions are identical with the analytical case to compare their results.

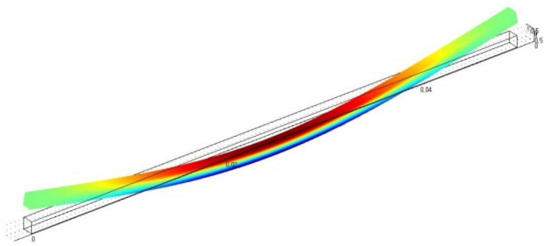

Figure 3. Finite element model for thermo-elastic analysis

Figs. 3~7 show the FEA results, and the peaks correspond to maximum TED. The worst thickness values obtained by FEA are summarized in Table 4, which presents the FEA results matches very well with the analytical ones.

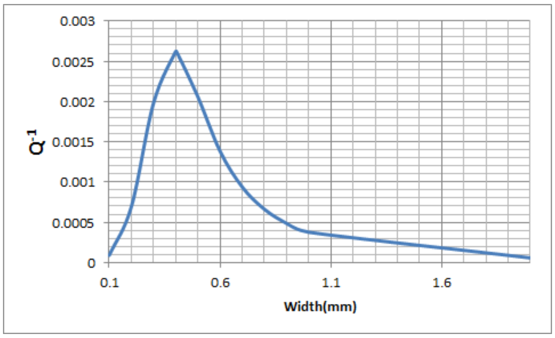

Figure 4. Thermo-elastic damping by finite element analysis for fixed-fixed condition

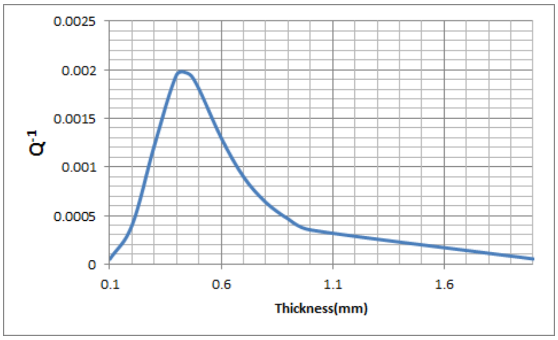

Figure 5. Thermo-elastic damping by finite element analysis for fixed-pinned condition

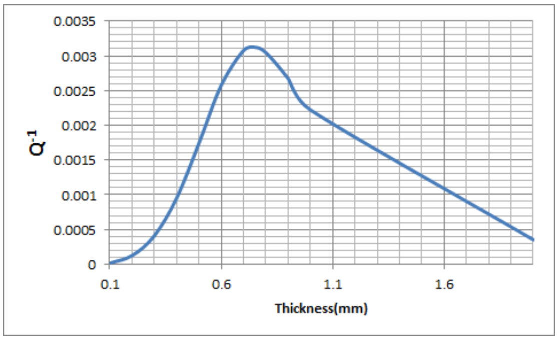

Figure 6. Thermo-elastic damping by finite element analysis for fixed-free condition

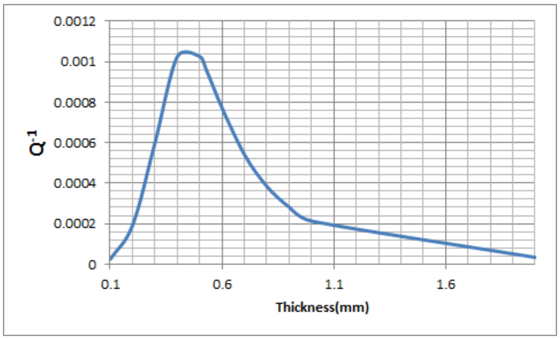

Figure 7. Thermo-elastic damping by finite element analysis for pinned-pinned condition

Table 4 Comparison of worst thicknesses for different boundary conditions

boundary condition	worst thickness (analytical)	worst thickness (FEM)	deviation
free-free,	0.404mm	0.4mm	1%
fixed-pinned,	0.458mm	0.458mm	0%
fixed-free	0.75mm	0.75mm	0%
pinned-pinned	0.53mm	0.5mm	6%

5. Conclusion

Thermo-elastic damping of a beam-type resonator was analyzed theoretically. The procedure to identify the worst thickness of the resonant beam for various boundary conditions in regard to the high Q factor is presented in this paper. Finite element analysis was also performed to numerically evaluate the thermo-elastic damping, and the results showed good consistency with the analytical ones.

TED has an important role for the design of a high Q metal resonator, which can be used in high-precision vibratory sensors. The methodology presented here can be extended to more complex resonant structures such as ring, cylindrical, or spherical types.

후기

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