직교모듈라격자의 멀티플라이어에 관하여
On Multipliers of Orthomodular Lattices

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요약

Orthomodular lattice is a mathematical description of quantum theory which is based on the family $CS(H)$ of all closed subspaces of a Hilbert space H. A partial multiplier is a function F from a non-empty subset D of a commutative semigroup A into A such that $F(x)y = xF(y)$ for every elements x, y in A. In this paper, we define the notion of multipliers on orthomodular lattices and give some properties of multipliers. Also, we characterize some properties of orthomodular lattices by multipliers.

1. Introductions

Quantum logic was introduced by G. Birkhoff and J. V. Neumann as a mathematical description for the quantum mechanics [1]. Orthomodular law and orthomodular lattices were studied to improve the quantum logic [2, 3].

An orthocomplementation on a bounded lattice L is a unary operation $'$ on L satisfying the following axioms.

(1) $a \le b$ implies $b' \le a'$,
(2) $a'' = a$,
(3) $a \vee a' = 1$ and $a \wedge a' = 0$.

An orthomodular lattice is a bounded lattice L with an orthocomplementation $'$ on L satisfying the orthomodular law : for any $a, b \in L$, $a \le b$ implies $a \vee (a' \wedge b) = b$.

The family $CS(H)$ of all close subspaces of a Hilbert space H gives rise to an orthomodular lattice [3].

In this paper, we define the notion of multipliers on orthomodular lattices and give some properties of this multiplier, and we characterize some properties of orthomodular lattices by multipliers.

2. Multipliers of Orthomodular Lattices.

A map φ from an orthomodular lattice L to itself is called a multiplier of L if $\varphi(a) \wedge b = a \wedge \varphi(b)$ for every $a, b \in L$.

Lemma 1. If φ is a multiplier of an orthomodular lattice L, then it has the following properties.

(1) $\varphi(a) \le a$ for every $a \in L$,
(2) $a \le b$ implies $\varphi(a) \le \varphi(b)$ for any $a, b \in L$,
(3) $\varphi(\varphi(a)) = \varphi(a)$ for every $a \in L$.

Example 2. Let L be a lattice and $x \in L$. If we define a map $\varphi_x : L \to L$ by $\varphi_x(a) = x \wedge a$ for every $a \in L$, then φ_x is a multiplier of L.

Theorem 3. If φ is a multiplier of an orthomodular lattice L, then it is a meet-homomorphism of L and $\varphi(a \wedge b) = \varphi(a) \wedge b = a \wedge \varphi(b)$.

The converse of Theorem 3 is not true in general, as the following example show.

Example 4. Let L be a orthomodular lattice with $|L| \ge 2$. The map f defined by $f(a) = 1$, for all $a \in L$, is a meet-homomorphism of L, but not multiplier, because for any a, $b \in L$ with $a \ne b$, $f(a) \wedge b = 1 \wedge b = b \ne a = a \wedge 1 = a \wedge f(b)$.

For any multiplier φ of an orthomodular lattice L, $Ker\varphi$, $Im\varphi$ and $Fix\varphi$ are the kernel, the image and the set of all fixed point of φ respectively, and for any subset X of L, we define $\downarrow X = \{a \in L \mid a \le x \text{ for some } x \in X\}$.

Lemma 5. Let φ be a multiplier of an orthomodular lattice L. Then it has the following properties.

(1) $Ker\varphi$ and $Im\varphi$ are subsemilattices of L as meet-semilattice,

(2) $\downarrow Ker\varphi = Ker\varphi$ and $\downarrow Fix\varphi = Fix\varphi$,

(3) $Fix\varphi = Im\varphi$.

The multiplier φ_x defined in Example 2 is called a simple multiplier of L.

Lemma 6. Let L be an orthomodular lattice. Then

(1) $\varphi_x(y) = \varphi_y(x)$ for every $x,\ y \in L$,

(2) $x \le y$ implies $\varphi_x(y) = x$ for any $x,\ y \in L$.

Theorem 7. An orthomodular lattice L is distributive if and only if the simple multiplier φ_x is a join-homomorphism for every $x \in L$.

Let $F(L)$ be the family of all functions from an orthomodular lattice L to itself. If we define a binary relation $\le$ on $F(L)$ by

$f \le g \Leftrightarrow f(a) \le g(a)\ (a \in L)$,

then $(F(L),\ \le)$ is a poset. Futhermore, for each $f,\ g \in F(L)$, define two functions $f \wedge g,\ f \vee g : L \rightarrow L$ by

$(f \wedge g)(a) = f(a) \wedge g(a),\ (f \vee g)(a) = f(a) \vee g(a)$

for every $a \in L$. Then $(F(L), \wedge, \vee)$ is a lattice.

Let $M(L)$ and $SM(L)$ be the families of all multipliers and simple multipliers, respectively, of L. Then $SM(L) \subseteq M(L) \subseteq F(L)$.

Theorem 8. Let L be an orthomodular lattice. If we define a map $\Phi : L \rightarrow M(L)$ by $\Phi(x) = \varphi_x$ for each $x \in L$, then Φ is order-embedding. That is, L is order- isomorphic to $\Phi(L) = SM(L)$.

The simple multipliers $\varphi_{x \wedge y}$ and $\varphi_{x \vee y}$ are a lower bound and an upper bound, respectively, of φ_x and φ_y by Theorem 8.

Lemma 9. Let L be an orthomodular lattice. Then for each $x, y \in L$,

(1) $\varphi_x \vee \varphi_y \le \varphi_{x \vee y}$,

(2) $\varphi_x \wedge \varphi_y = \varphi_{x \wedge y}$.

From Lemma 9(2), we know that $SM(L)$ is a sub-semilattice of $F(L)$, but $\varphi_x \vee \varphi_y \notin M(L)$ in general.

Theorem 11. Let L be an orthomodular lattice. Then L is distributive if and only if $\varphi_x \vee \varphi_y = \varphi_{x \vee y}$ for every $x,\ y \in L$, that is, $SM(L)$ is a sublattice of $F(L)$.

For any elements $a,\ b$ in a bounded lattice L with an orthocomplementation $'$, we say a commutes with b, in symbols aCb, if $a = (a \wedge b) \vee (a \wedge b')$. For any subset M of L, we define

$C(M) = \{a \in L \mid aCx \text{ for all } x \in M\}$,

in particular, we denote $C(x)$ for $C(\{x\})$.

Lemma 12. Let L be a bounded lattice with an ortho-complementation $'$. Then

$y \in C(x)$ if and only if $\phi_x(y) = y$.

Theorem 13. Let L be a bounded lattice with an ortho- complementation $'$. Then L is an orthomodular lattice if and only if $x \le y$ implies $\phi_x(y) = y$ for any $x, y \in L$.

Corollary 14. Let L be a bounded lattice with an ortho- complementation $'$. Then L is an orthomodular lattice if and only if $\uparrow x \subseteq C(x)$ for every $x \in L$, where $\uparrow x = \{y \in L \mid x \le y\}$.

■ References ■

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