A Comparison of the Rudin-Osher-Fatemi Total Variation model and the Nonlocal Means Algorithm

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Abstract

In this study, we compare two image denoising methods which are the Rudin-Osher-Fatemi total variation (TV) model and the nonlocal means (NLM) algorithm on medical images. To evaluate those methods, we used two well known measuring metrics. The methods are tested with a CT image, one X-Ray image, and three MRI images. Experimental result shows that the NML algorithm can give better results than the ROF TV model, but computational complexity is high.

***Keywords*:** Image denoising, total variation filter, nonlocal means filter.

I. INTRODUCTION

Image denoising is historically one of the oldest concerns in image and is still a necessary preprocessing step for many applications. The random noise is usually modeled by a probabilistic distribution. In many cases, a Gaussian distribution is assumed. However, some applications require more specific ones, like the gamma distribution for radar images (speckle noise) or the Poisson distribution for tomography. Unfortunately, it is usually impossible to identify the kind of noise involved for a given image. If no model of degradation is available, some assumptions have to be made. A commonly used model is the following. Let $u: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be an original image describing a real scene (the unknown), and let v be the observed image (the data) of the same scene (i.e., a degradation of u). We assume that

$$v = u + n, \qquad (1)$$

where n stands for a noise. Given v, the problem is then to reconstruct u knowing (1). This problem is ill posed, and we are able to carry out only approximation of u.

In this study, we compare two image denoising methods which are Rudin-Osher-Fatemi TV model and NLM algorithm.

The TV model was introduced by Rudin, Osher, and Fatemi [1] and Rudin and Osher [2]. They observe that if we minimize the TV norm of the image under some given conditions we will get a nonlinear diffusion filter. This idea gives a rigorous mathematical tool to introduce nonlinear diffusion filters and has been used as a regularization method for many applications where one needs to identify discontinuous functions. Motivated by the TV norm filter, many similar filters have been proposed in the literature, see ([4]-[11]).

The NLM algorithm was proposed by Buades, Coll, and Morel [3]. Compared to the previous denoising methods, the NLM algorithm measures the similarity based on the neighborhood of the pixel rather than the pixel value itself, which results in the further suppression of residual noise.

The remainder of this paper is organized as follows. Section II presents a mathematical background of the Rudin-Osher-Fatemi TV model and the NLM algorithm. In section III, the acquired results are illustrated. Finally, we conclude our paper in section IV.

II. IMAGE DENOISING METHODS

2.1. Method noise

Definition (Method noise) *Let u be an image and D_h a denoising operator depending on a filtering parameter h. Then, we define the* method noise *as the image difference*

$$u - D_h u.$$

The application of a denoising algorithm should not alter the non noisy images. So the method noise should be very small when some kind of regularity for the image is assumed. If a denoising method performs well, the method noise must look like a noise even with non noisy images and should contain as little structure as possible.

2.2. Rudin-Osher-Fatemi total variation model.

The original image $u(x, y)$ is supposed to have a simple geometric description, namely a set of connected sets, the objects, along with their smooth contours, or edges. The image is smooth inside the objects but with jumps across the boundaries. The functional space modeling these properties is $BV(\Omega)$, the space of integrable functions with finite TV

$$|u|_{BV} = \int_{\Omega} \sqrt{u_x^2 + u_y^2}$$

Given a noisy image $v(x,y)$, Rudin-Osher-Fatemi proposed to recover the original image $u(x,y)$ as the solution of the constrained minimization problem:

$$\mathrm{argmin}_u |u|_{BV} \tag{2}$$

subject to the noise constraints involving the mean

$\int_\Omega u dxdy = \int_\Omega v dxdy$

and standard deviation

$\int_\Omega \frac{1}{2}(u-v)^2 dxdy = \sigma^2$, where $\sigma > 0$ is given.

The preceding problem is naturally linked to the unconstrained problem

$$\mathrm{argmin}_u |u|_{BV} + \lambda \|v-u\|_{L^2}^2 \tag{3}$$

for a Lagrange multiplier λ. The solution of the (3) exists and is unique. The parameter λ controls the tradeoff between the regularity and fidelity terms. As λ gets smaller the weight of the regularity terms increases. Therefore λ is related to the degree of filtering of the obtained solution of the minimization problem. The Euler-Lagrange equation associated with the minimization problem is given by

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2}}\right) - \lambda(u-v) = 0 \text{ in } \Omega,$$

with $\frac{\partial u}{\partial n} = 0$ on the boundary of $\Omega = \partial\Omega$.

The solution procedure uses a parabolic equation with time as an evolution parameter, or equivalently, the gradient descent method. This means that

$$u_t = \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2+u_y^2}}\right)$$

$-\lambda(u-v)$, for $t > 0, x, y \in \Omega$, $\quad (4a)$

$u(x,y,0)$ given, $\quad (4b)$

$\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. $\quad (4c)$

As t increases, denoised version of image can be approached. The numerical method is given in [1].

2.3 NLM algorithm

Given a discrete noisy image $v = \{v(i) | i \in \Omega \subset \mathbb{R}^2\}$, the estimated value $NL[v](i)$ for a pixel i, is computed as a weighted average of all the pixels in the image,

$$NL[v](i) = \sum_{j\in\Omega} w(i,j)v(j),$$

where the family of weights $\{w(i,j)\}_j$ depend on the similarity between the pixels i and j, and satisfy the usual conditions $0 \le w(i,j) \le 1$ and $\sum_j w(i,j) = 1$.

The similarity between two pixels i and j depends on the similarity of the intensity gray level vectors $v(\mathcal{N}_i)$ and $v(\mathcal{N}_j)$, where $\mathcal{N}_k$ denotes a square neighborhood of fixed size and centered at a pixel k. This similarity is measured as a decreasing function of the weighted Euclidean distance, $\|v(\mathcal{N}_i) - v(\mathcal{N}_j)\|_{2,a}^2$, where $a > 0$ is the standard deviation of the Gaussian kernel. The application of the Euclidean distance to the noisy neighborhoods raises the following equality

$$E\|v(\mathcal{N}_i) - v(\mathcal{N}_j)\|_{2,a}^2 = \|u(\mathcal{N}_i) - u(\mathcal{N}_j)\|_{2,a}^2 + 2\sigma^2,$$

where σ^2 is the noise variance. This equality shows that the Euclidean distance preserves the order of similarity between pixels. So the most similar pixels to i in v also are expected to be the most similar pixels to i in u. The weights associated with the quadratic distances are defined by

$$w(i,j) = \frac{1}{Z(i)} e^{-\frac{\|v(\mathcal{N}_i)-v(\mathcal{N}_j)\|_{2,a}^2}{h^2}},$$

where $Z(i)$ is the normailizing factor

$$Z(i) = \sum_j e^{-\frac{\|v(\mathcal{N}_i)-v(\mathcal{N}_j)\|_{2,a}^2}{h^2}}$$

And the parameter h controls the decay of the exponential function, and therefore the decay of the weights, as a function of the Euclidean distance.

III. EXPERIMENTAL RESULTS

In this paper two well known measuring metrics are used to evaluate the effectiveness of two methods, which are introduced here briefly.

3.1 Signal to noise ratio (SNR)

Measuring the amount of noise by its standard deviation, $\sigma(n)$, one can define the signal to noise ratio (SNR) as

$$SNR = \frac{\sigma(u)}{\sigma(n)},$$

where $\sigma(u)$ denotes the empirical standard deviation of u,

$$\sigma(u) = \left(\frac{1}{NM}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\left(u(i,j) - \bar{u}(i,j)\right)^2\right)^{\frac{1}{2}},$$

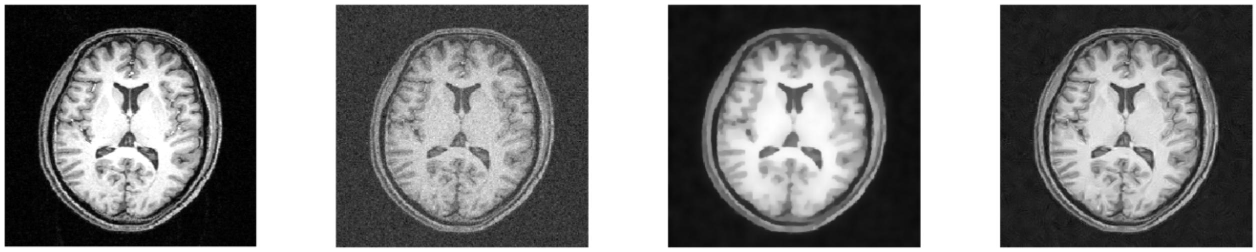
and $\bar{u}(i,j) = \frac{1}{NM}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1} u(i,j)$ is the average grey-level value. Here, when the denoised image has a large SNR it will be closer to the original image and will have a better quality.

3.2 Mean square error (MSE) metric

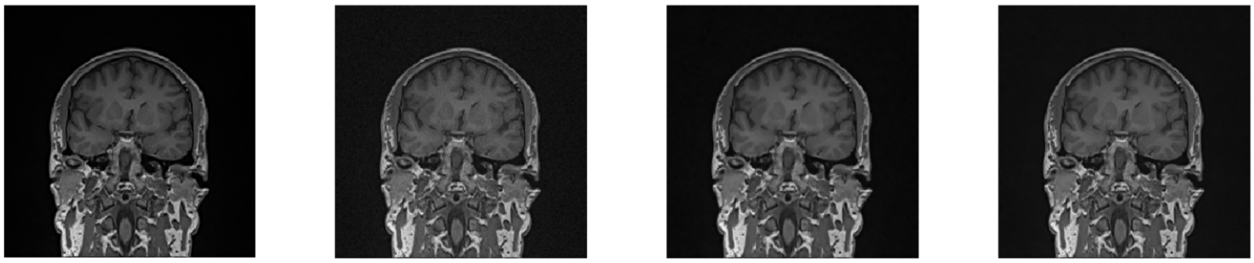
This metric measures the average of the squares of the “errors” and is defined as follows

$$MSE = \frac{1}{NM}\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}(v(i,j) - u(i,j))^2$$

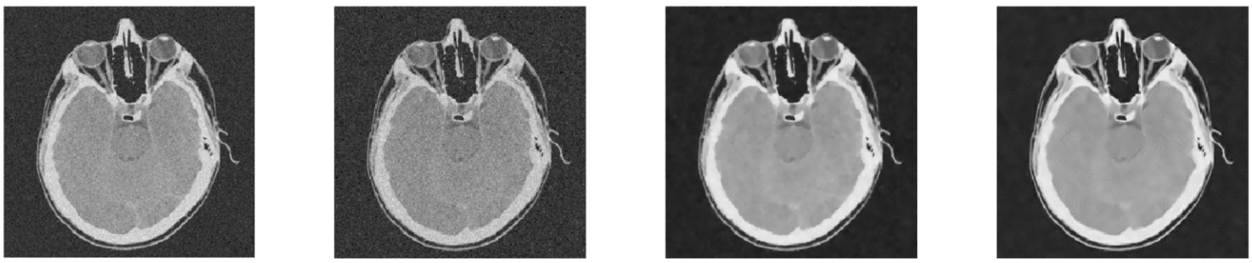
In this metric, the smaller the MSE value, the better is the denoising performance.



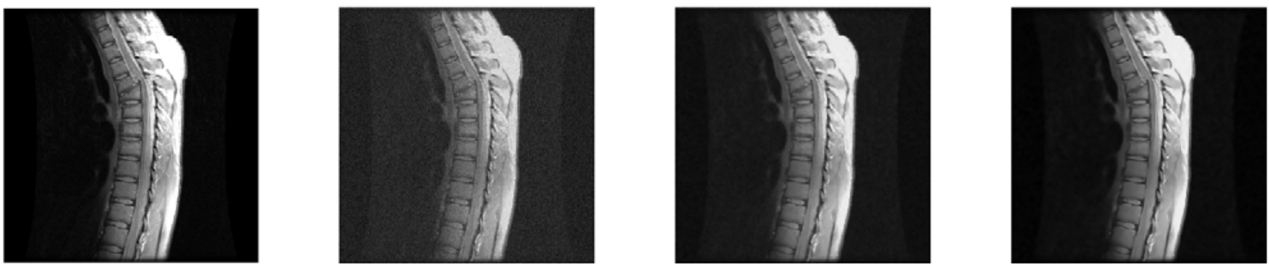
a) MRI Brain (Axial view) b) Noisy image, $\sigma = 0.09$ c) TV model d) NLM algorithm



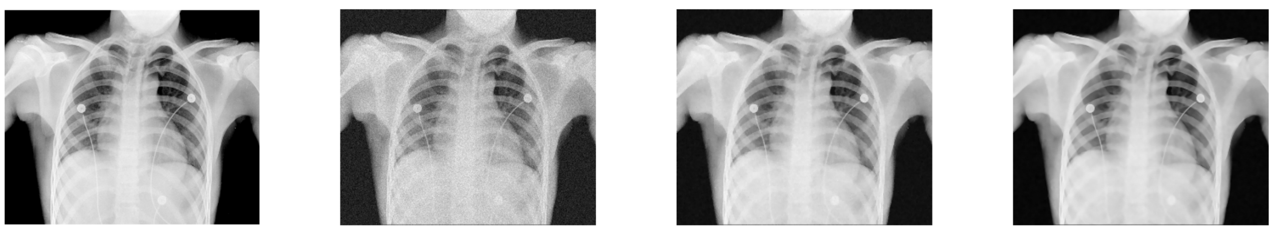
a) MRI Brain (Coronal view) b) Noisy image, $\sigma = 0.09$ c) TV model d) NLM algorithm



a) CT Head b) Noisy image, $\sigma = 0.04$ c) TV model d) NLM algorithm



a) MRI Spine b) Noisy image, $\sigma = 0.04$ c) TV model d) NLM algorithm



a) X-Ray Chest b) Noisy image, $\sigma = 0.04$ c) TV model d) NLM algorithm

Fig.1. Comparison of ROF TV model and NLM algorithm

In our experiment, we tested TV model and NLM methods on the five sample images and resultant images are shown in Fig.1.

	SNR		MSE	
	TV	NLM	TV	NLM
MRI brain (axial view)	34.33	**60.63**	0.0039	**0.0022**
MRI brain (coronal view)	227.78	**248.91**	0.0041	**0.0037**
CT head	25.84	**28.26**	0.0020	**0.0017**
MRI spine	266.64	**303.73**	2.21e-004	**1.93e-004**
X-Ray chest	528.60	**570.89**	2.54e-004	**2.33e-004**

Table1. Signal to noise ratio and Mean square error of ROF TV model and NLM algorithm.

As shown in Table 1, the SNR values of NLM algorithm are higher and MSE values are lower than that of ROF TV model on all sample images. The NLM algorithm is observed to obtain much better results compared to ROF TV model.

	TV	NLM
MRI brain (axial view)	**1.61**	11.02
MRI brain (coronal view)	**0.96**	11.34
CT head	**0.97**	10.96
MRI spine	**4.43**	43.98
X-Ray chest	**4.79**	49.59

Table2. Time (in seconds) taken for denoising images using ROF TV model and NLM algorithm.

It can be observed from the Table 2 that the computation time is much greater for the NLM algorithm than the ROF TV model.

IV. Conclusion

In this study we have applied ROF TV model and the NLM algorithm on medical images.

The NLM algorithm gives much better result compared to the ROF TV model. It can be seen from the measuring metrics SNR and MSE.

However, computation time had a noticeable difference in the two algorithms**.** Most of the computation time for the NLM algorithm was spent similarity calculation of local areas for each pixel.

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