Reweighted L1 Minimization for Compressed Sensing

*Hyuk Lee **Sunho Park ***Byonghyo Shim
Korea University
*2003210007@korea.ac.kr

Abstract

Recent work in compressed sensing theory shows that m×n independent and identically distributed sensing matrices whose entries are drawn independently from certain probability distributions guarantee exact recovery of a sparse signal with high probability even if m≪n. In particular, it is well understood that the L1 minimization algorithm is able to recover sparse signals from incomplete measurements. In this paper, we propose a novel sparse signal reconstruction method that is based on the reweighted L1 minimization via support recovery.

Ⅰ. Introduction

In many data acquisition paradigm, the relationship between the measurement and the natural signal is described by

$$y = \Phi x \qquad (1)$$

where x is the signal vector, Φ is the measurement matrix, and y is the measurement vector (sample). In order to reduce the necessary sampling rate, a powerful recovery algorithm is indispensable. In fact, due to the ability achieving reduction of the necessary sampling rate beyond the limit set by Nyquist, the compressed sensing (CS) has received much attention in recent year. CS is a recently introduced novel framework that goes against the traditional data acquisition paradigm. The central problem in CS is the recovery of a vector x from its linear measurements y of the form

$$y_i = \langle x, \varphi_i \rangle, \qquad 1 \leq i \leq m \qquad (2)$$

where m is assumed to be much smaller than n. When x is an n × 1 unknown vector with only k (k < m) nonzero components, a particular way of solving (1) which has recently generated a large amount of research is called 1-minimization [1]. It proposes solving the following problem

$$\min \| x \|_1 \qquad s.t. \quad y = \Phi x \qquad (3)$$

While L0 minimization is non-convex and combinatorial, 1-minimization is convex problem and practical for real applications. In spite of the substantial reduction in complexity, accuracy of the recovery algorithm is still a major problem since there is a difference between the L0 and L1 norms.

Solving under-determined systems by L1 minimization has a long history. It is at the heart of many numerical algorithms for approximation, compression, and statistical estimation. Rigorous results for L1 minimization began to appear in the late-1980', with Donoho and Stark [2] and Donoho and Logan [3]. Applications for 1-minimization in statistical estimation began in the mid-1990' with the introduction of the LASSO and related formulations [4], also known as Basis Pursuit [5], proposed in compression applications for extracting the sparsest signal representation from highly overcomplete frames. Around the same time other signal processing groups started using L1-minimization for the analysis of sparse signals [6].

In this paper, we introduce a robust sparse signal reconstruction method that provides savings in iterations, yet improves the reconstruction performance. By assigning the bigger weights to elements of x which are more likely to be zero, the proposed method improves the recovery accuracy. Owing to the direct benefit on performance, there have been a number of studies [7], [10].

Our method is distinct from these approaches. In order to assign the weight initially, additional support recovery method based on modified orthogonal matching pursuit (mOMP) is introduced. Due to the selection of the columns of Φ that is most strongly correlated with the residual, update of the index set depends more on the procedure of the least-square problem. At the least-square problem of the first iteration, we reduce the error of the signal approximation by exploiting the additional columns that is largely correlated with the measurement vector y.

This paper is structured as follows. After reviewing the background of the reweighted L1 minimization algorithm and OMP in Section II, we will present the method of the mOMP in Section III. The simulation results are provided in Section IV and we conclude in Section V.

Ⅱ. Background

The problem is that a system with fewer equations than unknowns usually has infinitely many solutions and thus, it is apparently impossible to identify which of these candidate solutions is indeed the correct one without some additional information. In many instances, however, the object we wish to recover is known to be structured in the sense that it is sparse or compressible. This means that the unknown object depends upon a smaller number of unknown parameters.

Under sparsity assumptions, one would want to recover a signal x, e.g., the coefficient sequence of the signal in the appropriate basis, by solving the combinatorial optimization problem

$$\min \| x \|_0 \quad s.t. \quad y = \Phi x \tag{4}$$

where x0 = |{i : xi = 0}|. This is of little practical use, however, since the optimization problem (4) is non-convex and generally impossible to solve as its solution usually requires an intractable combinatorial search. A common alternative is to consider the convex problem

$$\min \| x \|_1 \quad s.t. \quad y = \Phi x \tag{5}$$

where $\| x \|_1 = \sum_{i=1}^{n} |x_i|$. Unlike (4), this problem is convex and is solved efficiently. The programs (4) and (5) differ only in the choice of objective function, with the latter using an 1 norm as a proxy for the literal L0 sparsity count.

E. J. Candes considers one such alternative, which aims to help rectify a key difference between the L1 and L0 norms, namely, the dependence on magnitude: larger coefficients are penalized more heavily in the L1 norm than smaller coefficients, unlike the more democratic penalization of the L0 norm. To address this imbalance, a weighted formulation of L1 minimization is designed to more democratically penalize non-zero coefficients. Consider the weighted L1 minimization problem

$$\min \sum_{i=1}^{n} w_i |x_i| \quad s.t. \quad y = \Phi x \tag{6}$$

where w1, w2, · · · , wn are positive weights. In general, the weights are inversely proportional to the true signal magnitude, thus, it is of course impossible to construct the precise weights without knowing the signal itself. We introduce a reweighted L1 minimization based on the support recovery refers to the problem of detecting the support set. Although we do not know the original signal exactly, we are able to assign the weights initially by detecting the support set.

Ⅲ. The Modified Orthogonal Matching Pursuit

Since the vector x is k-sparse, the vector Φx belongs to one of L =nCk subspaces spanned by k of the n columns of Φ. Estimation of the support set is the selection of one of these subspaces. Mathematically, the ML estimator can be described as follows. Given a subset J ⊆ {1, 2, · · · , n}, let P_{Jy} denote the orthogonal projection of the vector y onto the subspace spanned by the vectors {aj |j ∈ J}. The ML estimate of the support set is

$$I_{ML} = \arg\max \| P_{Jy} \|^2 \tag{7}$$

where |J| denotes the cardinality of J. That is, the ML estimate is the set of k indices such that the subspace spanned by the corresponding columns of Φ contain the maximum signal energy of y. Since the number of subspaces, L, grows exponentially in n and k, an exhaustive search is computationally infeasible.

In our study, to detect the support set , we employ the mOMP method. Since x has only k non-zero components, the data vector y is a linear combination of k columns from Φ. We need to determine which columns of Φ participate in the measurement vector y. At each iteration, we choose the column of Φ that is most strongly correlated with the remaining part of y. Then we subtract off its contribution to y and iterate on the residual.

The OMP employs the least squares . Since the solution x of the first iteration has an error, the error is accumulated with the continuous iteration. In general, the index of the column that has large correlation is more likely to be support when the x has a fast decaying distribution. In our modified OMP method, we exploit these indices as a prior information.

Ⅳ. Simulation Result

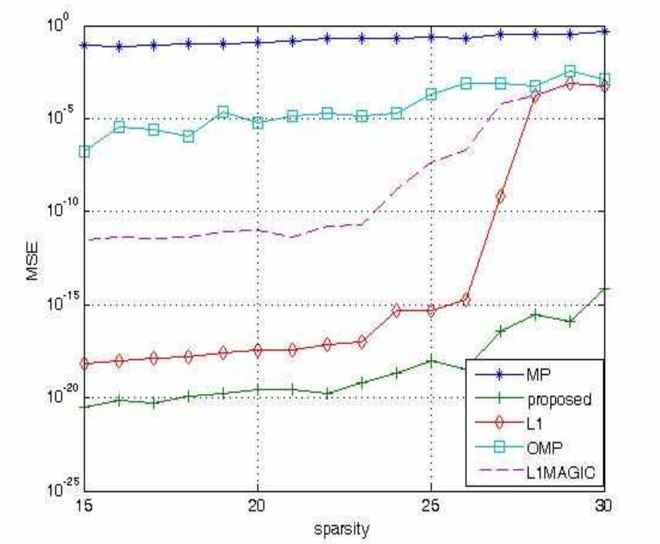

Fig. 1 : Sparse signal recovery from m=100 random measurements of a length n=256 signal.

In this section, we observe the performance of the proposed algorithm over various sparsity. We select a sparse signal x of length n = 256 The k non-zero spike positions are chosen randomly. We set m = 100 and sample a random m × n matrix Φ with i.i.d. Gaussian entries. As a measure for the performance, mean square error is used. As shown in Fig. 1, the matching pursuit (MP), unweighted L1, OMP, and the L1MAGIC are compared.

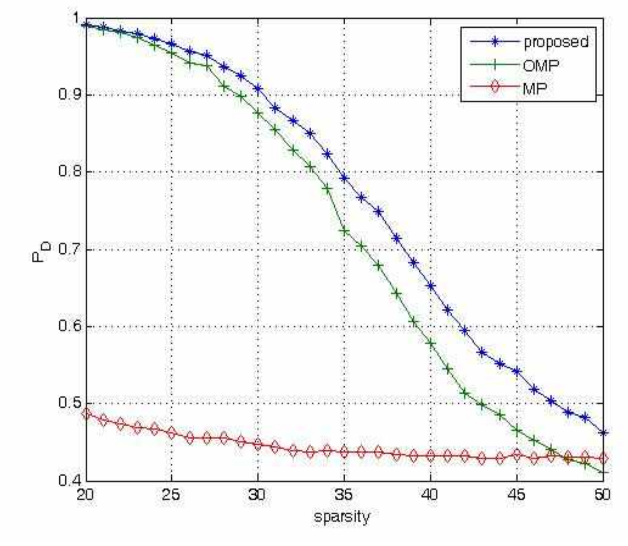

Fig. 2 : Recovery probability of the support of the sparse signal

In Fig. 1, the performance curves for the length n = 256 sparse signal are provided. The proposed algorithm yields lower mean square error than the other algorithms over almost entire sparsity range of interest. When the sparsity (the number of the non-zero value) is 20, the proposed algorithm gets a 1000 times better than the unweighted L1. Over the sparsity of 30, the performance achieved by the proposed algorithm dramatically improves.

Ⅴ. Further Directions

As shown in Fig. 1, we need to analyze the variation of the performance near the k = m/α. Since the proposed method is applied to noiseless version of L1 minimization, we need to compare the recovery performance of noisy version to noiseless.

Acknowledgement

This work was supported in part by mid-career Researcher Program through NRF grant funded by the MEST (No. 2009-0083945) and the second BK21 project.

Reference

[1] E. Candes and T. Tao, "Decoding by linear programming,"IEEE Trans. on Information Theory, 51(12), pp. 4203-4215, Dec. 2005.

[2] D. L. Donoho and P. B. Stark, "uncertainty principles and signal recovery,"SIAM J. Appl. Math. 49, no. 3, pp. 906-931, 1989.

[3] D. L. Donoho and B. F. Logan, "Signal recovery and the large sieve,"SIAM J. Appl. Math. 52, no. 2, pp. 57-591, 1992.

[4] R. Tibshirani, "Regression shrinkage and selection via the lasso,"J. Roy. Statist. Soc. Ser. B 58, no. 1, pp. 267-288, 1996.

[5] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit,"SIAM J. Sci. Comput. 1, pp. 33-61, 1998.

[6] M. S. O'rien, A. N. Sinclair, and S. M. Kramer, "Recovery of sparse spike time series by 1 norm deconvolution,"IEEE Trans. Signal Processing, vol. 42, pp. 3353-3365, 1994.

[7] E. J. Candes, M. B. Wakin, and S. Boyd, "Enhancing Sparsity by Reweighted 1 Minimization,"Journal of Fourier Analysis and Applications., 14(5), pp. 877-905, Dec. 2008.

[8] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, " Simple Proof of the Restricted Isometry Property for Random Matrices,"2008.

[9] J. A. Tropp and A. C. Gilbery, "Signal recovery from random measurements via orthogonal matching pursuit,"2006.

[10] W. Xu, M. A. Khajehnejad, A. S. Avestimehr, and B. Hassibi, "Breaking through the Thresholds: an Analysis for Iterative Reweighted 1 Minimization via the Grassmann Angle Framework," ICASSP 2010.