ASSESSMENT OF PROPERTY INTERPOLATION METHODS IN LEVEL SET METHOD

J.K. Park,*1 J.M. Oh2 and K.H. Kang3

레벨셋 기법의 물성 보간 방법에 대한 고찰

박 준 권,*1 오 정 민,2 강 관 형3

In level set method, material properties are made to change smoothly across an interface of two materials with different properties by introducing an interpolation or smoothing scheme. So far, the weighted arithmetic mean (WAM) method has been exclusively adopted in level set method, without complete assessment for its validity. We showed here that the weighted harmonic mean (WHM) method for rate constants of various rate processes, including viscosity, thermal conductivity, electrical conductivity, and permittivity, gives much more accurate results than the WAM method. The selection of interpolation scheme is particularly important in multi-phase electrohydrodynamic problems in which driving force for fluid flow is electrical force exerted on the phase interface. Our analysis also showed that WHM method for both electrical conductivity and permittivity gives not only more accurate, but also more physically realistic distribution of electrical force at the interface. Our arguments are confirmed by numerical simulations of drop deformation under DC electric field.

Key Words : 자유계면(Free Surface), 레벨셋 기법(Level Set Method), 물성 보간(Property Interpolation), 전기수력학(Electrohydrodynamics), 액적 변형(Drop Deformation)

1. INTRODUCTION

There are generally two types of numerical approaches for solving free surface problems. The first approach is front tracking where the interface is explicitly tracked. This includes the boundary integral method and some semi-Lagrangian front tracking methods such as marker-and-cell method (MAC) and vortex-in-cell method (VIC). The second approach is front capturing which includes volume-of-fluid method (VOF) and level set method (LSM). The level set formulation of the moving interface was introduced by Osher and Sethian in [1]. The level set method (LSM) can handle interface which is propagating with curvature dependent speed and topological merging and breaking, without explicitly tracking the interface.

In front capturing methods, multiphase-flow problems which possess a sharp phase interface are analyzed with a single equation without explicitly distinguishing the two phases. To facilitate the computation, the step change of material properties at the phase interface is smoothed out by introducing interpolation (or smoothing) schemes. The interpolation is applied across several nearest meshes with respect to phase interfaces.

A simple and the most popular method to interpolate material properties is taking the weighted arithmetic mean (WAM) of material properties of two phases ($\alpha_1$and α_1) as follows:

$$\alpha = \alpha_1(1-H) + \alpha_2 H \tag{1}$$

1 포항공과대학교 대학원 기계공학과
2 University of Twente, Netherland
3 포항공과대학교 기계공학과
* Corresponding author, E-mail: junkeun@postech.ac.kr

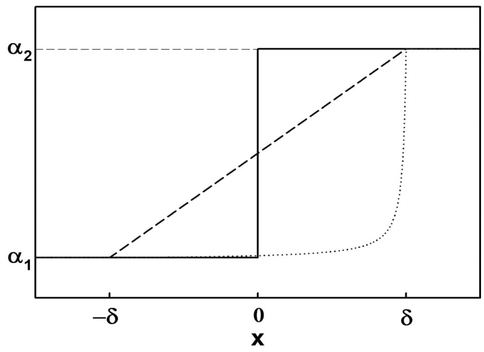


Fig. 1 Distribution of material property for the two interpolation schemes. Solid line is for sharp interface, dashed line is for WAM method, and dotted line is for WHM method, respectively

where $H = H(x)$ is the interpolation function, and x indicates the signed distance from a phase interface (see Fig. 1). The function H is typically set as 1/2 plus any anti-symmetric function of x. A simple form of H is, for example: $H=0$ if $x \le -\delta$ $H=(1+x/\delta)/2$ if $-\delta < x < \delta$ $H=1$ if $\chi \ge \delta$. Here, δ is the thickness of diffuse interface. For the WAM method, the distribution of α is anti-symmetric with respect to interface (x = 0). Thus, integration of the value of α itself in Eq. (1) along x direction is the same as that of unsmoothed (sharp) interfaces. Therefore, the WAM method will be certainly more suitable for parameters representing density of a transport quantity, such as mass density for momentum transport and thermal capacity for thermal energy transport. In particular, the WAM method ensures the conservation of mass for incompressible flows.

The material properties can be smoothed alternatively by using the weighted harmonic mean (WHM) method as follow

$$\frac{1}{\alpha} = \frac{1-H}{\alpha_1} + \frac{H}{\alpha_2}, \tag{2}$$

The smoothed material property is not anti-symmetric with respect to interface in this scheme (Fig. 1). Instead, the WHM method appears to reduce the impact of the greater material

property. The WHM can be viewed as the WAM of inverse of material property (α^{-1}). That is, integration of smoothed α^{-1} across the interface is the same as that of unsmoothed interface when using WHM.

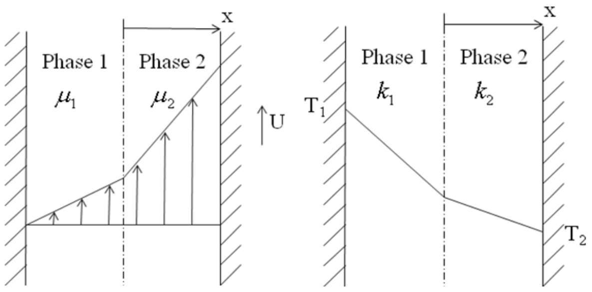
The interpolation method may somehow affect the overall accuracy of any numerical method which possesses a phase interface. In the level set method, the most popular interpolation scheme has hitherto been the WAM, probably because of its straightforwardness. For example, Sussman et al. [2], and Osher and Fedkiw [3] used the WAM interpolation method for all properties including density and viscosity. Many popular commercial packages including COMSOL Multiphysics, FLUENT, CFX, and STAR-CD employ the WAM as a interpolation method [4-7].

The LSM is often used to analyze the electrohydrodynamic problem involving drops due to its strength in topologically merging and breaking problems. In that case, the electricalforce should be evaluated by integrating the second derivatives of electrical potential across the potentially problematic phase interface. To our knowledge, there has been few systematic investigations which assessed the validity of the interpolation methods concerning the LSM. Only recently, Tomar et al. [9] compared the two methods for the case of electrical-conduction problem. They showed that inappropriate choice of interpolation method may significantly deteriorate the accuracy of computed electric field. They used the WHM method for electrical conductivity and permittivity, while other properties including density and viscosity are interpolated using the WAM method. For the finite volume method, Patankar [8] compared the performance of arithmetic mean with the harmonic mean for a conduction problem in composite materials. He suggested that harmonic mean should be used to interpolate diffusivity at cell face.

In this work, we compared the validity of the above-mentioned two methods in the LSM, and generalized the work of Tomar et al. [9]. We found out how the interpolation methods do affect the accuracy of the electrical force exerted on a phase interface. In section 2, we demonstrate the effect of interpolation in transport phenomena. In section 3, we investigate the consequences of interpolation on the electricalforce at interface. In section 4, a numerical simulation is performed for a submerged drop to support our arguments.

2. Validation of Interpolation Methods

First, we consider one-dimensional steady state transport processes which are exemplified by the fully developed Couette flow of two-layered fluids (Fig. 2(a)) and the steady conduction of heat across a composite wall (Fig. 2(b)). We introduce the coordinate variable x as shown in



(a) (b)

Fig. 2 (a) Fully developed two-layered flow; (b) heat transfer across a composite wall

Fig. 2. For the sake of simplicity, the material properties are assumed constant in each region, in original physical problems.

In a transport process, the rate equation is generally written as

$$J=-\alpha(x)\frac{d\phi(x)}{dx}, \tag{3}$$

where J is the generalized flux of a physical quantity, α the generalized diffusivity, and ϕ the generalized potential. In numerical simulation, the diffusivity is smoothed with some interpolation scheme, and therefore, the diffusivity is a function of position. Conservation of the physical quantity across the diffuse interface requires

$$\frac{dJ}{dx}=-\frac{d}{dx}\left(\alpha(x)\frac{d\phi(x)}{dx}\right)=S, \tag{4}$$

Here ϕ and α correspond to fluid velocity (u) and viscosity (μ) for fluid flows, temperature (T) and thermal conductivity (k) for heat conduction, and electrical potential (V) and electrical conductivity (σ) for electrical conduction, and so on. It is assumed that the source strength (S) i s null throughout the domain, The system we consider is linear, so that we can represent the transport process inside diffuse interface as an equivalent electrical circuit which is composed of electrical resistors in series as shown in Fig. 3.

Integration of Eq. (3) with respect to x yields,

$$\triangle\phi=\phi(\delta)-\phi(-\delta)=J\int_{-\delta}^{\delta}\frac{dx}{\alpha(x)}\ , \tag{5}$$

For an electrical system, $\triangle\phi$ and I = J × A correspond to voltage (V) and current, respectively, where A represents the surface area. In electrical analysis, the total resistance of the system (per unit area) becomes

$$R=\int_{-\delta}^{\delta}\frac{dx}{\alpha(x)}, \tag{6}$$

For step change of α, from α_1 to α_2,the true resistance of the system in Eq. (6) becomes

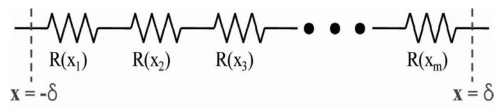


Fig. 3 Equivalent electrical circuit.

$$R_s=\delta\left(\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}\right), \tag{7}$$

For the WAM method, it becomes

$$R_{WAM}=\delta\left(\frac{2}{\alpha_1-\alpha_2}1n\left(\frac{\alpha_2}{\alpha_1}\right)\right), \tag{8}$$

The resistance of WHM method is identical to that of step change in Eq. (7) irrespective of choice of H. This means that the macroscopic response of the diffuse interface of WHM is the same as step change, for given $\triangle\phi$ or J. In other words, no error is induced in calculating the flux and the potential outside the diffuse interface, even though the sharp interface is replaced by a diffuse interface.

The resistance of the diffuse interface of WAM method relative to the true value is

$$\frac{R_{WAM}}{R_s}=\frac{2(\alpha_2/\alpha_1)1n(\alpha_2/\alpha_1)}{(\alpha_2/\alpha_1)^2-1}, \tag{9}$$

The deviation of the WAM method from the true value increases as α_2/α_1 increases (Fig. 4). Based on the above analytical results, we can conclude that WHM method is more relevant to interpolate the rate constant.

We carry out a numerical analysis for the two-layered Couette flow in Fig. 2, to compare the accuracy of the two methods. We chose the ratio of viscosity as 100, and

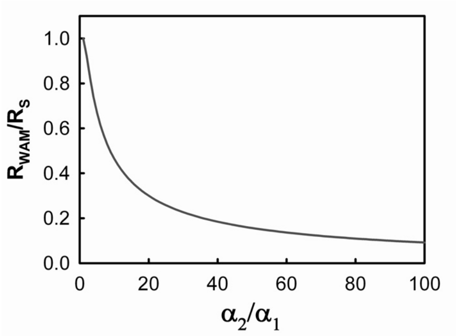
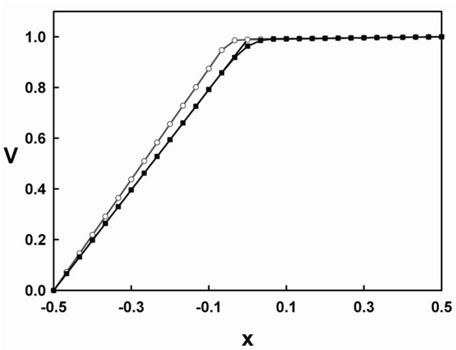
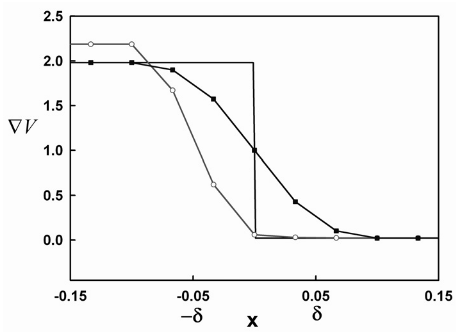


Fig. 4 The effect of diffusivity ratio on the equivalent resistance ratio of the diffuse interface using WAM and step change



(a)



(b)

Fig. 5 Effect of interpolation schemes on distribution of (a) velocity and (b) velocity gradient. Solid line indicates analytical solution. Blank symbol and solid symbol correspond to results using the WAM and the WHM, respectively

use 30 evenly spaced meshes. The thickness of the diffuse interface is 10% of domain length. The results for WAM method show a finite error in both velocity and velocity gradient, while WHM gives almost exact results except across the diffuse interface (Fig. 5). When we changed the number of meshes as 30, 60, and 120 with the thickness of the interface fixed, the errors in $\partial V/\partial \chi$ are 10.4%, 8.6%, 9.4% for WAM and zero for WHM (Table 1). The magnitude of error is of course proportional to deviation of resistance.

3. ELECTRICAL FORCE AT INTERFACE

In typical multi-phase electrohydrodynamic problems, driving force for fluid flows is electrical force exerted on the phase interface. In this case, selection of appropriate interpolation scheme is particularly important. The electrical force ($\mathbf{f_E}$) (per unit volume) is calculated by

$$\mathbf{f_E} = \rho_E \mathbf{E} - \frac{1}{2}|\mathbf{E}|^2 \nabla \varepsilon, \tag{10}$$

where ρ_E and $\mathbf{E}$ correspond to electrical charge density and electric field, respectively. To calculate the electrical force in Eq. (10) at an interface, we should reconsider the assumption of no flux generation at interface since the electrical charge at interface would generate flux (electric displacement).

When the fluids are dielectric, electrical potential is obtained by perfect dielectric model as follows:

$$\nabla \cdot (\varepsilon \nabla V) = 0, \tag{11}$$

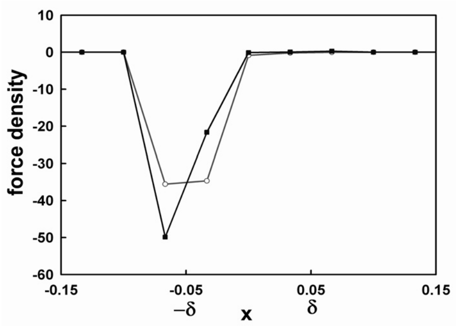
Since there is no electrical charge at the interface, the source strength is null. Therefore WHM method is

Table 1 Error of $\partial V/\partial x$ with respect to interface thickness

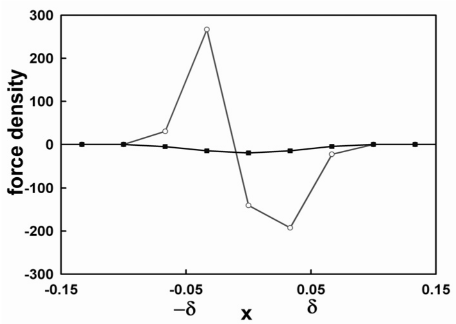
Mesh number	Error of $\partial V/\partial x$	
30	10.4%	0.0%
60	8.6%	0.0%
120	9.4%	0.0%

Table 2 Heat input (at 1 cycle for 1 cylinder)

σ	ε	**Error in total force**
WAM	WAM	21.9%
WAM	WAM	21.9%
WAM	WAM	0.0%
WAM	WAM	0.0%



(a)



(b)

Fig. 6 Effect of the electrical conductivity and permittivity interpolation schemes on the distribution of electrical force: (a) when the electrical conductivity is interpolated using WAM; (b) when the electrical conductivity is interpolated using WHM. Blank symbol and solid symbol correspond to permittivity interpolated using WAM and that interpolated using WHM, respectively

appropriate for the permittivity to calculate the electrical potential and force.

When the fluids are electrically conductive and satisfy $t^E << t^H$,where t^E and t^H indicate electrical charge relaxation time and hydrodynamic time, respectively, electrical conduction reach steady state much faster than time for fluid motion. Based on charge conservation law at steady state, electrical potential is calculated by leaky dielectric model as follows:

$$\nabla \cdot (\sigma \nabla V) = 0 , \qquad (12)$$

Since there is no electrical charge source, WHM method is appropriate for electrical conductivity to calculate the electrical potential and force. This analytical result implies an interesting fact that the calculated total electrical force

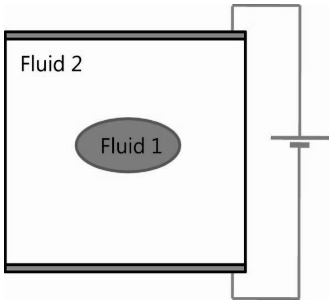
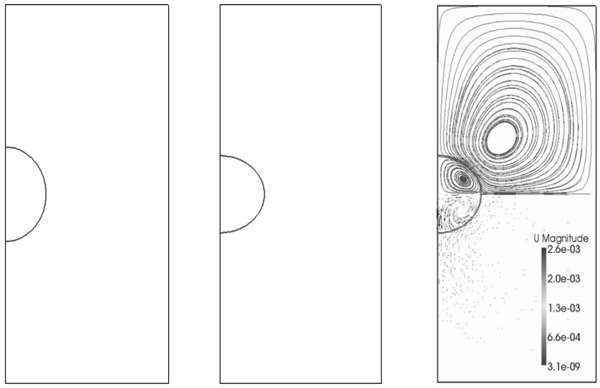


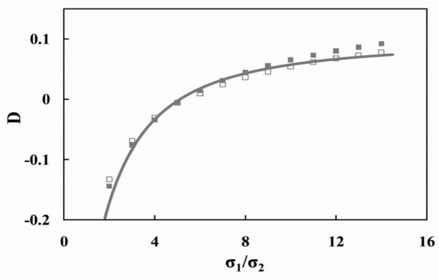
Fig. 7 Drop deformation under DC electric field



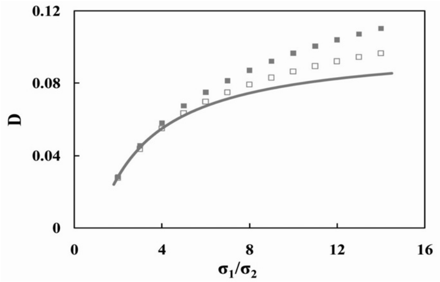
(a) (b) (c)

Fig. 8 Computational domain and results: (a) interface at initial state; (b) interface at equilibrium state; (c) velocity vectors and stream lines at equilibrium state

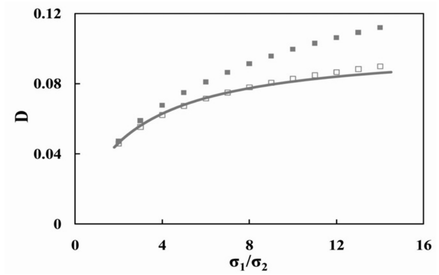
at interface of conductive fluids is independent of the interpolation scheme of permittivity, since there is no influence of permittivity in Eq. (12). We carried out a simple numerical calculation for one-dimensional steady state electrical conduction through two conductive fluids. We chose the ratios of electrical conductivity and permittivity as 100 and 50, respectively, and used 30 evenly spaced meshes. The thickness of the diffuse interface is 10% of domain length. Interpolation scheme of the permittivity influences only the 'distribution' of the electrical force in the diffuse interface without affecting the total electrical force as shown in Fig. 6 and Table 2. The electrical force using WAM for permittivity shows rather fluctuating distribution with some force of opposite direction, whereas WHM shows moderate and physically realistic behavior. Therefore we recommend using WHM interpolation scheme for both electrical conductivity and permittivity.



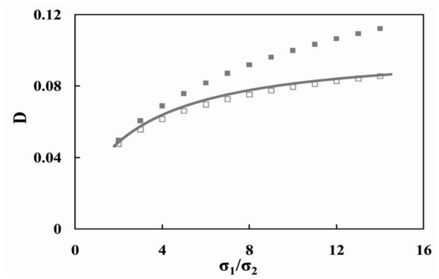
(a)



(b)



(c)



(d)

Fig. 9 Colmpari son of the drop deformation factor between Taylor theory and the computational results using WAM and WHM: (a) when the ratio of permittivity is 10, (b) 1, (c) 0.2 and (d) 0.1. Solid symbol and blank symbol correspond to the results using WAM and WHM, respectively

4. NUMERICAL VERIFICATION

We numerically simulated the deformation of a conducting drop under DC electric field to verify our argument (Fig. 7). We assume that the deformation is axially symmetric. The drop deformation factor, which is defined as $D=(L-B)/(L+B)$ is obtained, using the WAM and WHM methods for various ratios of material property. Here, L and B are the vertical and radial axis from end to end of the deformed drop, respectively. The results are compared with the following analytical solution of Taylor which is obtained by a perturbation analysis [10]:

$$D=\frac{9Ca_E}{16}\frac{f_d(\sigma_1/\sigma_2,\varepsilon_1/\varepsilon_2,\mu_1/\mu_2)}{(2+\sigma_1/\sigma_2)^2}, \tag{13}$$

$$f_d=\left(\frac{\sigma_1}{\sigma_2}\right)^2+1-2\frac{\varepsilon_1}{\varepsilon_2}+\frac{3}{5}\left(\frac{\sigma_1}{\sigma_2}-\frac{\varepsilon_1}{\varepsilon_2}\right)\frac{(2+3\mu_1/\mu_2)}{(1+\mu_1/\mu_2)} \tag{14}$$

$$Ca_E=E_\infty{}^2\varepsilon_2 R_d/\gamma \tag{15}$$

Where Ca_E, R_d and γ indicate electrical capillary number, radius of drop, and surface tension, respectively. The computation domain shown in Fig. 7 has horizontal and vertical length of 4 mm and 8 mm, respectively. The computation domain is discretized with 80 by 160 meshes. A spherical drop of 1 mm radius (Fig. 8(a)) deforms into either prolate or oblate spheroid depending on the ratios of electrical conductivity, permittivity, and viscosity (Fig. 8(b)). Velocity vectors and stream lines at equilibrium stateare shown in Fig. 8(c).

The resuts are shown in Fig. 9. In most cases, WHM showed much better agreement with Taylor's theory. This result confirms our argument that the WHM interpolation scheme is more relevant to interpolate both the electrical conductivity and permittivity for the phase interface of electrically conducting fluids.

5. CONCLUSION

We proposed that WHM method is adequate for all kinds of rate constants including viscosity, thermal conductivity, electrical conductivity and permittivity. By analytical calculation and physical consideration about the total resistance of the diffuse interface, we compared the accuracy of the WHM and WAM methods. The WHM method showed much better accuracy than the WAM. Our analysis also showed that the WHM method for both electrical conductivity and permittivity gave more accurate and physically realistic distribution of electrical force at interface of electrically conducting fluids. We carried out a numerical simulation about drop deformation under DC electric field, and results of the simulation confirmed our argument that the WHM interpolation scheme is more relevant to interpolate both the electrical conductivity and permittivity for the phase interface of electrically conducting fluids.

ACKNOWLEDGEMENT

This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (R0A-2007-000-20098-0, R01-2006-000-10657-0, and ROA-2008-000-20045-0).

REFERENCES

[1] 1988, Osher, S. and Sethian, J.A., "Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations," *J. Comput. Phys*., Vol.79, p.12.
[2] 1994, Sussman, M. et al., "A level set approach for computing solutions to incompressible two-phase flow," *J. Comput. Phys*., Vol.114, p.146.
[3] 2003, Osher, S. and Fedkiw, R., *Level set methods and dynamic implicit surfaces,* Springer, pp.227-238.
[4] COMSOL Multiphysics 3.4, www.comsol.com.
[5] FLUENT 6.3, www.ansys.com.
[6] CFX 10.0, www.ansys.com.
[7] STAR-CD 4.06, www.cd-adapco.com.
[8] 1978, Patankar, S.V., "A numerical method for conduction in composite materials, flow in irregular geometries and conjugate heat transfer," *International Heat Transfer Conference*, 6th, Toronto, Canada, p.297.
[9] 2007, Tomar, G. et al., "Two-phase electrohydrodynamic simulations using a volume-of-fluid approach," *J. Comput. Phys.*, Vol.227, p.1267.
[10] 1966, Taylor, G., "Studies in electrohydrodynamics I. The circulation produced in a drop by an electric field," *Proc. of Royal Society of London*, Series A, pp.159-166.