Noninformative Priors for Step Stress Accelerated Life Tests in Exponential Distribution

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Abstract　This paper deals with noninformative priors for such as Jeffres' prior, reference prior and probability matching prior for scale parameter of exponential distribution when the data are collected in multiple step stress accelerated life tests.

We find the noninformative priors for this model and show that the reference prior satisfies first order matching criterion. Using artificial data, we perform Bayesian analysis for proposed priors.

핵심주제어 : 계단 가속수명시험, 지수분포, 무정보적 사전분포, 베이지안 분석

Key Words : step stress accelerated life test, exponential distribution, noninformative priors, bayesian analysis

1. Introduction

In many reliability studies, the life tests were made under various environmental conditions. But for extremely reliable units it is in general impossible to make life tests under the usual conditions because the life times of units under the usual conditions may tend to be large and then the testing time may be very long. As a common approach to overcome this problem, the accelerated life tests (ALTs) are widely used, in which samples of units are subjected to conditions of greater stress than the usual conditions. For example, accelerated test conditions involve higher than usual temperature, voltage, pressure, vibration, cycling rate, load, etc. , or some combination of them.

The step stress ALT is commonly used in engineering practice. We interest the step stress ALT wherein the stress on unfailed units is allowed to change at preassigned times until they fail.

The existing literature on analysis of step-stress ALT centered around three types of models.

DeGroot and Goel (1979) proposed tampered random variables (TRV) model which the effect of changing the stress from s_1 to s_2 $(s_1 < s_2)$ is to multiply the remaining life of the unit at changing time τ (which they called it as tampering point) by some unknown factor, called tampering coefficient, $\alpha (0 < \alpha < 1)$. The proposed model is

$$Y = \begin{cases} X, & X \le \tau \\ \tau + \alpha(X - \tau), & X > \tau. \end{cases} \quad (1)$$

There is an another model for analyzing the accelerated life test data, which was proposed by Bhattacharyya and Soejoeti (1989). They

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assumed that the effect of changing the stress is to multiply the initial failure rate function $\lambda_1(y)$ by an unknown factor α subsequent to the change point τ. Denoting the failure rate function of the step-stress life length by $\lambda^*(y)$, the tampered failure rate (TFR) model is defined as

$$\lambda^*(y) = \begin{cases} \lambda_1(y), & y \le \tau \\ \alpha\lambda_1(y), & y > \tau. \end{cases} \tag{2}$$

Nelson (1980) proposed cumulative exposure (CE) model as follows:

Let F^* be the cumulative distribution function of the step-stress data which can be specified by $F_i(y) = F(y \mid s_i), i = 1,2,$ where $F(y \mid s_i)$ is the cumulative distribution function of life length under the constant stress setting s_i. Thus the CE model is defined by

$$F^*(y) = \begin{cases} F_1(y), & y \le \tau \\ F_2(\nu_1 + y - \tau), & y > \tau. \end{cases} \tag{3}$$

where ν_1 is defined to be the solution of the equation $F_2(\nu_1) = F_1(\tau)$.

Bhattacharyya and Soejoeti (1989) indicated that the TRV, CE and TFR models are identical in the sense that TRV can be expressed by the other models through reparameterization, when distribution under usual stress is exponential distribution.

For applications of ALT model to the real data, multiple (a model with more than two tampering points) step-stress ALT model will be applicable to the extremely reliable items. There have been several works extending two step-stress (or simple step-stress) ALT model to multiple step-stress model. Typical examples are Shaked and Singpurwalla (1983) and Madi (1993).

The papers mentioned above except Degroot and Geol (1976) solved the estimation problem in step-stress ALT models using the classical or nonparametric methods.

From a Bayesian point of view, DeGroot and Goel (1979) studied the Bayesian estimation of parameters and optimal design of the model (1) when the lifetime under use stress is exponential distribution. They considered two independent gamma priors for parameter estimation. Kim, Lee and Kang (2006) developed noninformative priors for two step stress ALT model. They developed that Jeffreys' prior, reference prior, probability matching prior for the scale parameter in two step stress ALT when the lifetime under normal stress distributes as exponential distribution.

In Bayesian analysis, owing to the lack of prior knowledge about parameters or lack of time to accumulate the information about the model, there may be an inevitable situation to use noninformative priors. The most commonly used noninformative prior is Jeffreys' (1961) prior, which is proportional to the positive square root of the determinant of the Fisher information matrix. Jeffreys' prior plays a major role in many one parameter models, but Jeffreys' prior frequently runs into serious difficulties in the presence of nuisance parameters. Usually, Jeffreys' prior does not match frequentist coverage probability.

In recent years, many efforts have been done for finding noninformative priors such as reference or probability matching prior in Bayesian analysis. There has been a great deal of studies for finding noninformative priors.

Welch and Peers (1963), Peers (1965) and Stein (1985) found a prior which requires the frequentist coverage probability of the posterior region of a real-valued parametric function to match the normal level with a remainder of $o(n^{-1/2})$, where n is the sample size. Tibshirani (1989) reconsidered the case when the real valued parameter of interest is orthogonal to the nuisance parameter vector. These priors, as usually referred to as 'first order' matching priors, were further studied in Datta and Ghosh (1995a, 1995b, 1996).

Recently, Ghosh and Mukerjee (1997) developed a 'second order', that is, $o(n^{-1})$, matching prior. They extend the finding in Mukerjee and Dey (1993) to the case of multiple nuisance parameters based on quantiles, and also develop a second order matching prior based on distribution function.

On the other hand, Ghosh and Mukerjee (1992), and Berger and Bernardo (1989,1992) extended Bernardo's (1979) reference prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion described earlier.

In this paper, we will generalize two step-stress TRV model (1) to multiple step-stress TRV model. For the Bayesian analysis, we derive the reference prior and matching prior for the scale parameter when the lifetime distribution under normal stress is exponential. Through the orthogonal transformation (Cox and Reid (1987)), we first find the orthogonal reparameterization for scale parameter and then find reference prior and matching prior. We show that the proposed matching prior is the first order matching prior and that there exists no second order matching for multiple step-stress ALT model. We show that, under the proposed noninformative priors, the joint posterior for the parameters is proper. And some simulation results and example are given.

2. Step-Stress Accelerated Life Test Model

Consider the realistic situation of accelerated life testing where we continue increasing the stress level on the unfailed items over a preassigned number $k(\geq 1)$ of times. And we assume that the lifetime distribution under normal stress follows exponential distribution with parameter theta of which the probability density function (pdf) is given by

$$f_1(x \mid \theta) = \theta e^{-\theta x}, 0 < x < \infty, 0 < \theta < \infty. \quad (4)$$

When $k \geq 2$, we call it as a multiple step-stress ALT model. Now, we generalize the simple step-stress TRV model (1) to a multiple step-stress TRV model. Let $0 < \tau_1 < \tau_2 < \cdots < \tau_k < \infty$ be the k's tampering points.

Starting to the normal stress level s_0, at the tampering point τ_1, we raise stress level to $s_1 (> s_0)$, and so on. According to the stress level s_i, $i = 1,2,\cdots,k$, there is tampering coefficient α_i which represents the effect of stress change. Let Y be the lifetime under multiple step-stress pattern, then, the multiple step-stress TRV model can be described as follows.

$$Y = \begin{cases} X, \text{ if } X \leq \tau_1, \\ \dfrac{X - \sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1}}{\prod_{j=0}^{l-1}\alpha_j^{-1}} + \tau_{l-1}, \\ \quad \text{if } \sum_{i=1}^{l-1}\dfrac{\tau_i - \tau_{i-1}}{\prod_{j=0}^{i-1}\alpha_j} < X \leq \sum_{i=1}^{l}\dfrac{\tau_i - \tau_{i-1}}{\prod_{j=0}^{i-1}\alpha_j}, \\ \quad l = 2,\cdots,k, \\ \dfrac{X - \sum_{i=1}^{k}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1}}{\prod_{j=0}^{k}\alpha_j^{-1}} + \tau_k, \\ \quad \text{if } X > \sum_{i=1}^{k}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1}, \end{cases} \quad (5)$$

where $\alpha_0 \equiv 1$, $\tau_0 \equiv 0$, $0 < \alpha_i < 1, i = 1,\cdots,k$ and X is exponentially distributed random variable with density (4).

If we denote $F_1(y \mid \theta)$ as the distribution function of X, then the distribution function F of Y is given by

$$F(y \mid \theta, \underline{\alpha}) = \begin{cases} F_1(y \mid \theta), y \leq \tau_1 \\ F_1\left(\sum_{i=1}^{l-1}\dfrac{(\tau_i - \tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \dfrac{(y - \tau_{l-1})}{\prod_{j=0}^{l-1}\alpha_j} \middle| \theta\right), \\ \quad \tau_{l-1} < y \leq \tau_l, \quad l = 2,\cdots,k, \\ F_1\left(\sum_{i=1}^{k}\dfrac{(\tau_i - \tau_{i-1})}{\prod_{j=0}^{i-1}\alpha_j} + \dfrac{(y - \tau_k)}{\prod_{j=0}^{k}\alpha_j} \middle| \theta\right), \\ \quad y > \tau_k, \end{cases} \quad (6)$$

where $\underline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_k)$.

The probability density function $f(y \mid \theta, \underline{\alpha})$ of Y is given by

$$f(y \mid \theta, \underline{\alpha}) = \begin{cases} \theta \exp\{-\theta y\}, & y \le \tau_1 \\ \theta\left(\prod_{j=0}^{l-1} \alpha_j^{-1}\right) e^{-\theta\left[\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1} + (y-\tau_{l-1})\prod_{j=0}^{l-1}\alpha_j^{-1}\right]}, & \tau_{l-1} < y \le \tau_l, l = 2,\cdots,k, \\ \theta\left(\prod_{j=0}^{k} \alpha_j^{-1}\right) e^{-\theta\left[\sum_{i=1}^{k}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1} + (y-\tau_k)\prod_{j=0}^{k}\alpha_j^{-1}\right]}, & y > \tau_k, \end{cases} \tag{7}$$

where $0 < \theta < \infty$ and $0 < \alpha_i < 1, i = 1,2,\cdots,k.$

Let

$$\delta_1 = I(y \le \tau_1),$$

$$\delta_l = I(\tau_{l-1} < y \le \tau_l), l = 2,\cdots,k,$$

and

$$\delta_{k+1} = I(y > \tau_k),$$

where I is indicator function. Then $\sum_{i=1}^{k+1}\delta_i = 1.$ And the likelihood function per one observation y is given by,

$$L(\theta, \underline{\alpha} \mid y) = \theta\left(\prod_{l=1}^{k}\alpha_l^{-\sum_{i=l+1}^{k+1}\delta_i}\right) \times \exp\left\{-\theta y\delta_1 - \theta\sum_{l=2}^{k+1}\delta_l\left[\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1} + (y - \tau_{l-1})\prod_{j=0}^{l-1}\alpha_j^{-1}\right]\right\}. \tag{8}$$

The log-likelihood function for one observation is

$$\ell(\theta, \underline{\alpha} \mid y) = \log\theta - \sum_{l=2}^{k}\log\alpha_l \sum_{i=l+1}^{k+1}\delta_i - \theta y\delta_1 - \theta\sum_{l=2}^{k+1}\delta_l\left[\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})\prod_{j=0}^{i-1}\alpha_j^{-1} + (y - \tau_{l-1})\prod_{j=0}^{l-1}\alpha_j^{-1}\right]. \tag{9}$$

Usually, one purpose of the ALTs is the information about the parameter under the normal stress level. In our multiple step-stress TRV model, θ, which is the failure rate at the normal stress, is more important parameter than the others.

Now, we consider the reparameterization of the original parameters to accomplish the parameter orthogonality in the sense of Cox and Reid (1987). To do this, let

$$\omega_1 = \theta$$

and

$$\omega_l = \frac{\theta}{\prod_{j=1}^{l-1}\alpha_j}, \; l = 2,\cdots,k+1.$$

Then the log-likelihood function under the above reparameterization is given by

$$\ell(\underline{\omega} \mid y) = \sum_{l=1}^{k+1}\delta_l\log\omega_l - \omega_1 y\delta_1 - \sum_{l=2}^{k+1}\delta_l\left[\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})\omega_i + (y - \tau_{l-1})\omega_l\right], \tag{10}$$

where $\underline{\omega} = (\omega_1,\cdots,\omega_{k+1}) \in \Theta_\omega$, and $\Theta_\omega = \{\underline{\omega} \mid 0 < \omega_1 < \omega_2 < \omega_3 < \cdots < \omega_{k+1} < \infty\}$.

From the above reparameterized log-likelihood function (10), one can find the Fisher information matrix for $\underline{\omega}$. Let I_ω be the Fisher information matrix of $\underline{\omega}$. Then the information matrix I_ω is a diagonal matrix with elements I_i, $i = 1,2,\cdots,k+1$. The elements are given by

$$I_1 = \frac{1}{\omega_1^2}(1 - \exp\{-\omega_1\tau_1\}),$$

$$I_l = \frac{1}{\omega_l^2}exp\left(-\sum_{i=1}^{l-1}(\tau_i - \tau_{i-1})\omega_i\right), \; l = 2,3,\cdots,k, \times(1 - \exp\{-(\tau_l - \tau_{l-1})\omega_l\})$$

and

$$I_{k+1} = \frac{1}{\omega_{k+1}^2}exp\left(-\sum_{i=1}^{k}(\tau_i - \tau_{i-1})\omega_i\right).$$

3.Noninformative Priors

In this section, we will derive the noninformative priors in multiple step-stress TRV model. From the information matrix I_ω, one can find Jeffrey's prior for $\underline{\omega}$ as follows.

$$\pi^{J}(\underline{\omega}) \propto \left(\prod_{l=1}^{k+1}\omega_l^{-1}\right) \times \left[\left(\prod_{l=1}^{k}(1-\exp-(\tau_l-\tau_{l-1})\omega_l)\right) \times \left(\prod_{l=2}^{k+1}\exp-\sum_{i=1}^{l-1}(\tau_i-\tau_{i-1})\omega_i\right)\right]^{1/2}, \tag{11}$$

where $\underline{\omega} \in \Theta_\omega$. Using the identity

$$\sum_{l=2}^{k+1}\sum_{i=1}^{l-1}(\tau_i-\tau_{i-1})\omega_i = \sum_{l=1}^{k}(k-l+1)(\tau_l-\tau_{l-1})\omega_l$$

Jeffrey's prior can be rewritten as

$$\pi^{J}(\underline{\omega}) \propto \prod_{l=1}^{k+1}\omega_l^{-1} \times \prod_{l=1}^{k}(1-\exp\{-(\tau_l-\tau_{l-1})\omega_l\})^{1/2} \times \exp\left\{-\frac{1}{2}\left(\sum_{l=1}^{k}(k-l+1)(\tau_l-\tau_{l-1})\omega_l\right)\right\}.$$

It is well known that the prior (11) does not meet the nominal level coverage probability in case of the presence of nuisance parameters. To remedy this problem, we will find a reference prior and a matching prior when ω_1 is a parameter of interest.

We introduce the brief concept of a probability matching prior. For a prior π, let $\theta_1^{1-\alpha}(\pi;Y)$ be a percentile of the posterior distribution of θ_1, that is,

$$P^{\pi}\{\theta_1 \le \theta_1^{1-\alpha}(\pi;Y) \mid Y\} = 1-\alpha. \tag{12}$$

We want to find priors which satisfy

$$P^{\pi}\left\{\theta_1 \le \theta_1^{1-\alpha}(\pi;Y) \mid \theta,\alpha\right\} = 1-\alpha+o(n^{-\frac{u}{2}}) \tag{13}$$

for some $u>0$, as n goes to ∞. Priors π satisfying (13) are called a probability matching priors. If $u=1$, then π is called a first order matching prior, if $u=2$, π is called a second order matching prior.

Now, we want to find a matching prior when the parameter of interest is ω_1. Denote $\omega^{(2)}=(\omega_2,\cdots,\omega_{k+1})$ as a nuisance parameters. Based on the work of Tibshirani (1989), the first order probability priors, when the parameter of interest is ω_1, is given by,

$$\pi^{M}(\underline{\omega}) \propto \omega_1^{-1}(1-\exp\{-\omega_1\tau_1\})^{1/2}d(\omega^{(2)}), \tag{14}$$

where $d(\cdot)$ is an arbitrary differentiable function in its arguments. Clearly, the Jeffrey's prior (11) is not the first order matching prior.

Berger and Bernardo (1992a) developed the algorithm to find a reference prior. And Datta and Ghosh (1995) proposed the method of developing reference priors when the orthogonality of the parameters is satisfied. In our case, ω_1 is of more inferential importance than $\omega^{(2)}$, the one at a time reference prior is given by

$$\pi^{R}(\underline{\omega}) \propto \left(\prod_{i=1}^{k+1}\omega_i\right)^{-1} \times \left[\prod_{i=1}^{k}(1-\exp-\omega_{i}(\tau_i-\tau_{i-1}))\right]^{1/2}, \tag{15}$$

where $\underline{\omega}$ in Θ_ω. One can find the fact that this prior is also the first order matching prior.

4. Numerical Example

Using the noninformative priors developed in the previous section, we perform Bayesian analysis using real data.

The following 15 data, given by Proschan (1963), are a part of time intervals of successive failures of the air conditioning system in Boeing jet airplanes.

74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326

We assume that the time between successive failures is independent and exponentially distributed with parameter θ.

We apply the data to the model (5) with $k=2$. We assume that the acceleration factors $\alpha_1=0.1$ and $\alpha_2=0.1$. And the tempering point $\tau_1=50$ and $\tau_2=75$. Here, we choose τ_2 such that $\tau_1+(\tau_2-\tau_1)\alpha_1^{-1}=300$. The original data are changed as follows.

X	δ_1	δ_2	δ_3	Y
74	0	1	0	52.40
57	0	1	0	50.70
48	1	0	0	48.00
29	1	0	0	29.00
502	0	0	1	77.02
12	1	0	0	12.00
70	0	1	0	52.00
21	1	0	0	21.00
29	1	0	0	29.00
386	0	0	1	75.86
59	0	1	0	50.90
27	1	0	0	27.00
153	0	1	0	60.30
26	1	0	0	26.00
326	0	0	1	75.26

Using the above data, we compute the MLE and various Bayes estimates for the parameters. The estimates are given in the following table.

	MLE	π^J	π^M	π^R
θ	0.0118	0.0080	0.0080	0.0080
α_1	0.2159	0.1917	0.1454	0.1693
α_2	0.0573	0.0792	0.0679	0.0887

The estimated values are not quite different. All the estimates give overestimated values for θ and α_1, but α_2 is underestimated.

From the above table, we see that Bayes estimates give better estimates than MLE. For parameter α_1, the Bayes estimate under π^M is the closest to 0.1 whereas for α_2, the Bayes estimate under π^R is the closest to 0.1. And we can conclude that the overall performance of the Bayes estimates are superior to MLE.

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