A note on interval-valued functionals of random sets.

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Abstract

In this paper, we consider interval probability as a unifying concept for uncertainty and Choquet integrals with respect to a capacity functional. By using interval probability, we will define an interval-valued capacity functional and Choquet integral with respect to an interval-valued capacity functional. Furthermore, we investigate Choquet Choquet weak convergence of interval-valued capacity functionals of random sets.

Key words: random sets, interval probability, interval-valued capacity functional, Choquet integrals, Choquet weak convergence.

1. Interval probabilities and Choquet integrals

Throughout the paper, $R$ is the set of real numbers and

$$\mathcal{I}(R) = \{[a, b] | a, b \in R \text{ and } a \leq b\}.$$  

Then a element in $\mathcal{I}(R)$ is called an interval number. On the interval number set, we define: for each pair $[a, b], [c, d] \in \mathcal{I}(R)$ and $k \in R$, $$[a, b] + [c, d] = [a + c, b + d],$$ $$[a, b] \cdot [c, d] = [a \cdot c \wedge a \cdot d \wedge b \wedge c \wedge b \wedge d, a \cdot c \vee a \cdot d \vee b \vee c \vee b \vee d].$$

$$k[a, b] = \begin{cases} [ka, kb], & k \geq 0 \\ [kb, ka], & k < 0 \end{cases}$$

$[a, b] \leq [c, d]$ if and only if $a \leq c$ and $b \leq d$.

We note that $(\mathcal{I}(R), d_H)$ is a metric space, where $d_H$ is the Hausdorff metric defined by

$$d_H([a, b], [c, d]) = \max\{\sup_{x \in [a, b]} \inf_{y \in [c, d]} |x - y|, \sup_{y \in [c, d]} \inf_{x \in [a, b]} |x - y|\}$$
for all $[a, b], [c, d] \in \mathcal{I}(R)$. Then it is easy to see that for $[a, b], [c, d] \in \mathcal{I}(R)$,

$$d_H([a, b], [c, d]) = \max\{b - c, d - a\}.$$  

Let $X$ be a locally compact, second-countable, Hausdorff (LCSCH) space. Let $\Phi$ be the class of all closed sets in $X$, $\Psi$ the class of all compact sets in $X$, $O$ the class of all open sets in $X$, $\Phi_0 = \Psi - \{\varnothing\}$, and $B(X)$ the class of all Borel sets in $X$. We note that $B(\Phi)$ is the $\sigma$-algebra classes a generated by $\Phi^K$, $K \in \Psi$ and $\Phi_G$, $G \in O$, where $\Phi^K = \{F \in \Phi | F \cap K = \varnothing\}$ and $\Phi_G = \{F \in \Phi | F \cap G = \varnothing\}$.

Definition 2.1 (\cite{6}) Let $(\Omega, \Sigma, P)$ be a probability space and $(\Phi, B(\Phi))$ an measurable space.

(1) A measurable mapping $S: (\Omega, \Sigma, P) \rightarrow (\Phi, B(\Phi))$ is called a random closed set.

(2) The probability measure $Q$ induced on $B(\Phi)$ is defined by for each $B \in B(\Phi)$, $Q(B) = P(S^{-1}(B))$.

We note that the distribution of a
random set $S$ is uniquely determined by its hitting functional $T_S$ on $\Psi$ such that

$$T_S(K) = P\left((w | S(w) \cap K = \emptyset)\right), K \in \Psi.$$ 

From this, it is easy to see that $T_S$ satisfies the following properties (see [5]):

1. $T_S$ is upper semi-continuous on $\Psi$.
2. $T_S(\emptyset) = 0$ and $0 \leq T_S \leq 1$.
3. $T_S$ is monotone increasing on $\Psi$ and for $K_1, K_2, \ldots, K_n \in \Psi, n \geq 2$,

$$T_S(\bigcap_{j=1}^{n} K_j) \leq \sum_{\emptyset \neq \mathcal{J} \subset \{1, \ldots, n\}} (-1)^{\mathcal{J}+1} T_S(\bigcup_{j \in \mathcal{J}} K_j).$$

**Definition 2.2** ([6]) An interval-valued set function $\overline{P} (\cdot)$ on $\Sigma$ is called an interval probability if

1. $\overline{P}(A) = [L(A), U(A)]$,
2. $0 \leq L(A) \leq U(A) \leq 1$, $\forall A \in \Sigma$.
3. $\{A \in \Sigma | L(A) \leq P(A) \leq U(A)\} \neq \emptyset$.

**Definition 2.3** The interval probability measure $\overline{Q}$ induced on $B(\Phi)$ is defined by for each $B(\Phi), \overline{Q}(B) = \overline{P}(S^{-1}(B))$.

**Definition 2.4** (1) For every random closed set $S$, an interval-valued mapping $\overline{T}_S$ is said to be an interval-valued capacity functional if there exist two capacity functionals $T^1_S$ and $T^2_S$ such that $\overline{T}_S = [T^1_S, T^2_S]$.

(2) Let $f \in C(X)$. The Choquet integral of $f$ with respect to $\overline{T} = [T^1, T^2]$ is defined by

$$\left(\int fd\overline{T}\right) = \left[\int fdT^1, \int fdT^2\right].$$

(3) The sequence of interval-valued capacity functionals $\overline{T}_n$ $d_H$-converges in the Choquet weak sense to the interval-valued capacity functional $\overline{T}$ in $X$ if

$$\left(\int fd\overline{T}_n \rightarrow d_H \left(\int fd\overline{T}\right), \forall f \in C(X)\right)$$

whenever $\overline{T}_n \rightarrow c_\infty \overline{T}$ as $n \rightarrow \infty$.

**Theorem 2.5** Let $\overline{T}_n = [T^1_n, T^2_n]$ be a sequence of interval-valued capacity functionals and $\overline{T} = [T^1, T^2]$ an interval-valued capacity functional. Then $\overline{T}_n \rightarrow d_H c_\infty \overline{T}$ as $n \rightarrow \infty$ if and only if $T^i_n \rightarrow c_\infty T^i$ for $i = 1, 2$ as $n \rightarrow \infty$.

**Theorem 2.6** Let $\overline{T}_n = [T^1_n, T^2_n]$ be a sequence of interval-valued capacity functionals. If $\overline{T}_n \rightarrow d_H c_\infty \overline{T}$ as $n \rightarrow \infty$, then we have

$$d_H \left(\limsup_{n \rightarrow \infty} \overline{T}_n(B)\right) \leq \overline{T}(B), \forall B \in \Phi.$$