Existence of Solutions for the Semilinear Fuzzy Integrodifferential Equations

준선형 퍼지적분미분방정식에 대한 해의 존재성

권영철[1], 김미주[1], 이부영[1]

동아대학교 자연과 학대학 수학과
yckwun@dau.ac.kr, juju-bar@hanmail.net, bylee@dau.ac.kr

Abstract

This paper is to investigate the existence theorem for the semilinear fuzzy integrodifferential equation in E_N by using the concept of fuzzy number whose values are normal, convex, upper semicontinuous and compactly supported interval in E_N. Main tool is successive iteration method.

Keywords : Semilinear Fuzzy Integrodifferential Equations, fuzzy number, successive iteration

1. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva [3] studied the existence and uniqueness of solution for the fuzzy differential equation on E^n where E^n is normal, convex, upper semicontinuous and compactly supported fuzzy sets in R^n. Seikkala [7] proved the existence and uniqueness of fuzzy solution for the following equation:

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number in E^1. Diamond and Kloeden [2] proved the fuzzy optimal control for the following system:

$$\dot{x}(t) = a(t)x(t) + u(t), \quad x(0) = x_0,$$

where $x(\cdot), u(\cdot)$ are nonempty compact interval-valued functions on E^1. Kwun and Park [4] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E_N^1 using by Kuhn-Tucker theorems. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently, Park, Park and Kwun [6] find the sufficient condition of nonlocal controllability for the semilinear fuzzy integrodifferential equation with nonlocal initial condition.

In this paper, we study the existence theorem for the following semilinear fuzzy integrodifferential equation:

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] \quad (1)$$
$$+ f(t, x(t)) \, t \in I = [0, T],$$
$$x(0) = \phi_0 \in E_N, \quad (2)$$

where $A : I \to E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α-level intervals, $f : I \times E_N \to E_N$ is nonlinear continuous functions, $G(t)$ is $n \times n$ continuous matrix such that $\frac{dG(t)x}{dt}$ is continuous for $x \in E_N$ and $t \in I$ with $\|G(t)\| \le k, k > 0$.

2. Preliminaries

A fuzzy subset of R^n is defined in terms of membership function which assigns to each point $x \in R^n$ a grade of membership in the fuzzy set.

Such a membership function $m : R^n \to [0, 1]$ is used synonymously to denote the corresponding fuzzy set.

Assumption 1. m maps R^n onto $[0, 1]$.

Assumption 2. $[m]^0$ is a bounded subset of R^n.

Assumption 3. m is upper semicontinuous.

Assumption 4. m is fuzzy convex.

We denote by E^n the space of all fuzzy subsets m of R^n which satisfy assumption 1-4; that is, normal,

fuzzy convex and upper semicontinuous fuzzy sets with bounded supports.

In particular, we denote by E^1 the space of all fuzzy subsets m of R which satisfy assumption 1-4.

A fuzzy number a in real line R is a fuzzy set characterized by a membership function m_a as $m_a : R \rightarrow [0,1]$.

A fuzzy number a is expressed as $a = \int_{x \in R} m_a(x)/x$, with the understanding that $m_a(x) \in [0,1]$ represent the grade of membership of x in a and $\int$ denotes the union of $m_a(x)/x$'s [5].

Let E_N be the set of all upper semicontinuous convex normal fuzzy number with bounded α-level intervals.

This means that if $a \in E_N$ then the α-level set.

$$[a]^\alpha = \{x \in R : m_a(x) \geq \alpha, \ 0 < \alpha \leq 1\}$$

is a closed bounded interval which we denote by

$$[a]^\alpha = [a_l^\alpha, a_r^\alpha]$$

and there exists a $t_0 \in R$ such that $a(t_0) = 1$. (see[4])

The support Γ_a of a fuzzy number a is defined, as a special case of level set

by the following

$$\Gamma_a = \{x \in R : m_a(x) > 0\}$$

Two fuzzy numbers a and b are called equal $a = b$, if $m_a(x) = m_b(x)$ for all $x \in R$. It follows that

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha \ \ for \ all \ \ \alpha \in (0,1]$$

A fuzzy number a maybe decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha[a]^\alpha$$

where $\alpha[a]^\alpha$'s with α ranging from 0 to 1.

We denote the supremum metric d_∞ on E^n and the supremum metric H_1 on $C(I : E^n)$.

Definition 2.1 Let $a, b \in E^n$

$$d_\infty(a,b) = sup\{d_H([a]^\alpha, [b]^\alpha) : \alpha \in (0,1]\}$$

Where d_H is the Hausdorff distance.

Definition 2.2 Let $x, y \in C(I : E^n)$

$$H_1(x,y) = sup\{d_\infty(x(t), y(t)) : \ t \in I\}$$

Let I be a real interval. A mapping $x : I \rightarrow E$ is called a fuzzy process.

We denote

$$[x(t)]^\alpha = [(x_l^\alpha)'(t), (x_r^\alpha)'(t)], \ \ t \in I, \ 0 < \alpha \leq 1$$

The derivative $x'(t)$ of a fuzzy process x is defined by

$$[x'(t)]^\alpha = [(x_l^\alpha)'(t), (x_r^\alpha)'(t)], \ \ 0 < \alpha \leq 1$$

provided that is equation defines a fuzzy $x'(t) \in E_N$

The fuzzy integral

$$\int_a^b x(t)dt, \ \ a, b \in I$$

is defined by

$$\left[\int_a^b x(t)dt \right]^\alpha = \left[\int_a^b x_l^\alpha(t)dt, \int_a^b x_r^\alpha(t)dt \right]$$

provided that the Lebesque integral on the right exist.

Definition 2.3 (Puri and Ralescu [4]) The integral of a fuzzy mapping $F : [0,1] \rightarrow E^n$ is defined level-wise by

$$\left[\int_T F(t)dt \right]^\alpha = \int_T F_\alpha(t)dt$$
$$= \left\{ \int_T f(t)dt : f : T \rightarrow R^n \right.$$
$$\left. \text{ is a measurable selection for } F_\alpha \right\}$$

for all $\alpha \in [0,1]$.

Definition 2.4 A mapping $F : T \rightarrow E^n$ is bounded if there exists a constant $M > 0$ such that $d_\infty(F(t), \{0\}) \leq M$ for all $t \in T$.

It was proved by Puri and Ralescu [4] that a strongly measurable and integrably bounded mapping $F : T \rightarrow E^n$ is integrable.(i.e, $\int_T F(t)dt \in E^n$)

Theorem 2.1 If $F : T \to E^n$ is continuous then it is integrable.

Theorem 2.2 Let $F, G : T \to E^n$ be integrable and $\lambda \in R$. Then

(1) $\int_T (F(t) + G(t))dt = \int_T F(t)dt + \int_T G(t)dt$

(2) $\int_T \lambda F(t)dt = \lambda \int_T F(t)dt$

(3) $d_\infty(F, G)$ is integrable.

(4) $d_\infty(\int_T F(t)dt, \int_T G(t)dt \leq \int_T d_\infty(F(t), G(t))dt$

Definition 2.5 [1] The fuzzy process $x : I \to E_N$ is a solution of equations (1)-(2) without the inhomogeneous term if and only if

$$(\dot{x}_l^\alpha)(t) = \min\left\{ A_l^\alpha(t)\left[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds\right], \ i,j = l,r \right\},$$

$$(\dot{x}_r^\alpha)(t) = \max\left\{ A_r^\alpha(t)\left[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds\right], \ i,j = l,r \right\},$$

and

$$(x_l^\alpha)(0) = x_{0l}^\alpha, \qquad (x_r^\alpha)(0) = x_{0r}^\alpha.$$

Now we assume the following:

(H1) $S(t)$ is a fuzzy number satisfying, for $y \in E_N$ and $S'(t)y \in C^1(I : E_N) \cap C(I : E_N)$, the equation

$$\frac{d}{dt}S(t)y = A\left[S(t)y + \int_0^t G(t-s)S(s)yds\right]$$

$$= S(t)Ay + \int_0^t S(t-s)AG(s)yds, \ t \in I,$$

such that

$$[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)],$$

and $S_i^\alpha(t)$ $(i = l, r)$ is continuous. That is, there exists a constant $c > 0$ such that $|S_i^\alpha(t)| \leq c$ for all $t \in I$ and $S(0) = I$.

(H2) The inhomogeneous term $f : I \times E_N \to E_N$ is continuous function and satisfies a global Lipschitz condition.

$$d_H([f(s,x(s))]^\alpha, [f(y,y(s))]^\alpha) \qquad (3)$$
$$\leq c_2 d_H([x(s)]^\alpha, [y(s)]^\alpha)$$

for all $x(\cdot), y(\cdot) \in E_N$ and a finite constant $c_2 > 0$, $cc_2 < 1$ and $f(s, \{0\})$ is bounded on $C(I : E_N)$.

3. Existence theorems

In this section, we prove the existence and uniqueness of fuzzy solutions for the equation (1)-(2).

The (1)-(2) is related to the following fuzzy integral equation :

$$x(t) = S(t)\phi_0 + \int_0^t S(t-s)f(s,x(s))ds \quad (4)$$

$$x(0) = \phi_0 \in E_N \qquad (5)$$

where $S(t)$ is satisfy (H1).

Theorem 3.1 Suppose that hypotheses (H1)-(H2) are satisfied. Then there exists a unique solution $x(t)$ of the equation (1)-(2) on $C(I : E_N)$ and the successive iteration

$$x_0(t) = S(t)\phi_0 \qquad (6)$$

$$x_{n+1}(t) = x_0(t) + \int_0^t S(t-s)f(s,x_n(s))ds \quad (7)$$

are uniformly convergent to $x(t)$ on $C(I : E_N)$.(n= 0,1,2, ...)

References

[1] P. Balasubramaniam and S. Muralisankar, *Existence and uniqueness of fuzzy solution for semilinear fuzzy integrodifferential equations with nonlocal conditions*, International J. Computer & Mathematics with applications, 47 (2004), 1115–1122.

[2] P. Diamand and P.E. Kloeden, *Metric space of Fuzzy sets*, World Scientific (1994).

[3] O. Kaleva, *Fuzzy differential equations*, Fuzzy set and Systems 24 (1987), 301–317.

[4] Y.C. Kwun and D. G. Park, *Optimal control problem for fuzzy differential equations*, Proceedings of the Korea-Vietnam Joint Seminar (1998), 103–114.

[5] M. Mizmoto and K. Tanaka, *Some properties of fuzzy numbers*, Advances in Fuzzy Sets Theory and applications, North-Holland Publishing Company (1979), 153–164.

[6] J.H. Park, J.S. Park and Y.C. Kwun, *Controllability for the semilinear fuzzy integrodifferential equations with nonlocal conditions*, Lecture Notes in Artificial Intelligence 4223 (2006), 221–230.

[7] S. Seikkala, *On the fuzzy initial value problem*, Fuzzy Sets and Systems 24 (1987), 319–330.