Multi-scale coherent structures and their role in the energy cascade in homogeneous isotropic turbulence

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Abstract: In order to investigate the physical mechanism of the energy cascade in homogeneous isotropic turbulence, we introduce Galilean-invariant energy and its transfer rate in the real space as a function of position, time and scale. By using a database of direct numerical simulations (DNS) of homogeneous isotropic turbulence, it is shown that (i) fully developed turbulence consists of multi-scale coherent vortices of tubular shapes, (ii) the energy at each scale is mainly confined in vortex tubes with the radii of the same order of the length scale, and (iii) the energy transfer takes place around pairs (especially, anti-parallel pairs) of such vortex tubes. Based on these observations, it is suggested that the energy cascade can be caused, in the real space, by the process of the stretching and creation of smaller (i.e. thinner) vortex tubes by the straining field around pairs of larger (i.e. fatter) vortex tubes. Indeed, it is quite easy to find such events (in our DNS fields) which strongly support this scenario of the energy cascade.

Keywords: Energy cascade, Homogeneous isotropic turbulence, Direct-numerical simulation, Coherent structures

1. INTRODUCTION

Many of turbulence theories are based on the Kolmogorov similarity hypothesis[1]: that is, at high enough Reynolds numbers, turbulent fluid motions at small scales are statistically independent of large-scale structures, which must depend on the boundary conditions and/or external forcing sustaining the turbulence. This implies that small-scale statistics of turbulence at high Reynolds numbers are universal. The consequences of this hypothesis (the universality of the energy spectrum in high wave numbers, for example) have been well supported experimentally and numerically.

It is the energy cascade that gives the basis of this Kolmogorov hypothesis. The picture of the energy cascade is well described by the famous verse by Richardson[2]:

Big whirls have little whirls
which feed on their velocity,
and little whirls have lesser whirls,
and so on to viscosity —

which reads that the energy supplied to turbulence at a large scale (i.e. the integral length L) transfers to smaller and smaller scales, and it dissipates by the molecular viscosity at the smallest scale (i.e. the Kolmogorov length η). It is of importance that the energy transfers scale by scale. The information of the largest scale may well be forgotten through this energy cascade process, and therefore the small-scale universality can be achieved. Indeed, the Fourier analysis of turbulent velocity fields has been shown to support the energy cascade picture[3–5].

However, it has been an important open question what kind of physical process is relevant to the energy cascade. In the present study (the details of which have been published in Ref.[6]), we discuss this naive problem (i.e. the energy cascade in the real space) by the use of the database of the direct numerical simulations (DNS) of homogeneous isotropic turbulence.

2. SCENARIO OF THE ENERGY CASCADE

The analyses of DNS data shown in the next section suggest that the energy cascade in homogeneous turbulence can be caused by the physical process described as follows.

The energy supplied to a large scale (L) is possessed by vortex tubes with large radii of O(L). When a pair (especially, an anti-parallel pair) of these vortex tubes encounter, a strong straining region is created around the pair. In this straining region, vortices are stretched and smaller (more precisely, thinner) vortex tubes are created. In other words, the energy possessed by the fatter (L) vortex tubes transfers to smaller scales (L', say) by the process of vortex stretching. Thus created vortex tubes with radii of the smaller scale (L') confine the energy inside them. However, if a pair (especially, an anti-parallel pair) of them encounter, a strong straining region at the scale (L') is created around the pair, and further smaller-scale (i.e. further thinner, L'') vortex tubes are stretched and created. Thus the energy transfers from L' to L''. By several steps of similar processes, the smallest-scale (the Kolmogorov length) vortex tubes are created finally. Then, the energy possessed by vortex tubes with radii of O(η) is dissipated by the molecular viscosity in strong straining regions around them.

Incidentally, it is well known by the previous DNS studies[7–10] that energy is predominantly dissipated around the vortex tubes with radii of the order of the Kolmogorov length. This means that the proposed picture of the energy cascade at the scales in the inertial range $(\eta < \ell < L)$ have some analogy to the energy dissipation process at the Kolmogorov length.

Although the above scenario is different essentially from naive pictures such that large scale vortices break up to smaller ones by some instabilities, there have been many related studies which have mentioned the physical-space energy cascade in terms of vortex stretching (e.g.[11–13]). It is also worth mentioning the physical picture of the inverse energy cascade in two-dimensional turbulence, where the vortex stretching is inhibited. It is shown in Ref.[14] that the vortex thinning (instead of the stretching) in larger-scale straining regions leads to the inverse energy cascade.

3. NUMERICAL VERIFICATIONS

In this section, we show the DNS analyses which support the scenario of the energy cascade described in the preceding section. The analyzed DNS data of statistically homogeneous isotropic stationary turbulence in a periodic cube have been simulated by the de-aliased Fourier spectral method. (The details of the numerical scheme are given in Ref.[15].) The Reynolds number (based on the Taylor length) of the turbulent flow treated in the followings is about 187. The grid width, the

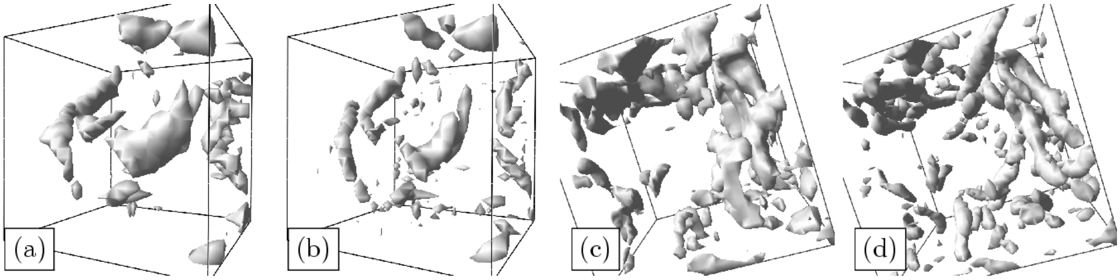

Fig. 1 (a,c) Iso-surfaces of the internal energy. (b,d) Iso-surfaces of the coarse-grained enstrophy. (a,b) $\ell = 18\eta$. The threshold of the iso-surfaces is the mean m plus the standard deviation σ. The size of shown box is $(1.1L)^3 \approx (200\eta)^3$. (c,d) $\ell = 74\eta$. The threshold of the iso-surfaces is m + 1.5σ. The box size is $(3.5\ L)^3 \approx (600\eta)^3$.

typical length scale of the external forcing, and the period of the boundary condition are about 2.3η, 410η and 1200η, respectively.

3.1 Galilean invariance and Lagrangian viewpoint

Since the energy cascade is one of the most important features of turbulence, its physical mechanism has been investigated by many authors. The wavelet analysis (see Ref.[16] for a review) has been employed frequently for this purpose. However, in this study, we use a simpler method constructed by paying attentions to the following two points.

The first point is the Galilean invariance. The phenomenon of the energy cascade itself must be Galilean invariant, but the cascading quantity, i.e. the energy which is the squared velocity, is not Galilean invariant in general. This implies that we have to define the energy which is independent of the frame of reference. Therefore, we employ the notion of the internal energy (defined in the thermodynamics) to introduce Galilean-invariant energy of fluid particles in an arbitrary size of cube.

The second is the fact that the dynamics of turbulence at high Reynolds number cannot be described in the frame fixed in the laboratory (i.e. from the Eulerian viewpoint). This was the difficulty encountered by the closure theories[17], and it is known that we need to introduce a Lagrangian viewpoint to overcome it. Therefore, in the followings, we carefully define the energy transfers in the frame of reference moving with fluid particles. Note that the importance of the Lagrangian viewpoint in discussions of turbulent energy cascade has been emphasized also by other authors[18, 19].

3.2 Internal energy and its transfer rate

First, we introduce the energy at scale ℓ, position $\boldsymbol{x}$ and time t as follows. Since the following definition of this scale-dependent energy is the analogy of the internal energy in the thermodynamics, we call it the internal energy of fluid motion. We take the cube $V(\boldsymbol{x},\ell)$ with the side ℓ centered at $\boldsymbol{x}$, and decompose the kinetic energy per unit mass

$$K(\boldsymbol{x},t\,|\,\ell) = \frac{1}{2\ell^3}\int_{V(\boldsymbol{x},\ell)} |\,\boldsymbol{u}(\boldsymbol{x}',t)\,|^2\ d\boldsymbol{x}' \tag{1}$$

into the translational energy

$$E(\boldsymbol{x},t\,|\,\ell) = \frac{1}{2}\left|\left\langle \boldsymbol{u}(\boldsymbol{x},t)\right\rangle_\ell\right|^2 \tag{2}$$

and the internal energy

$$\widetilde{U}(\boldsymbol{x},t\,|\,\ell) = \frac{1}{2\ell^3}\int_{V(\boldsymbol{x},\ell)} \left|\boldsymbol{u}(\boldsymbol{x}',t) - \left\langle \boldsymbol{u}(\boldsymbol{x},t)\right\rangle_\ell\right|^2 d\boldsymbol{x}' \tag{3}$$

Note that $K = E + \widetilde{U}$. Here,

$$\left\langle \boldsymbol{u}(\boldsymbol{x},t)\right\rangle_\ell = \frac{1}{\ell^3}\int_{V(\boldsymbol{x},\ell)} \boldsymbol{u}(\boldsymbol{x}',t)\,d\boldsymbol{x}' \tag{4}$$

is the mean velocity of fluid particles inside the cube, and called the translational velocity. Since $\widetilde{U}(\boldsymbol{x},t\,|\,\ell)$ is the internal energy of fluid motions at the scales smaller than ℓ, the energy of fluid motions at the scales between ℓ and ℓ/α (α is a constant; α = 2 is used in the followings) is expressed as

$$U(\boldsymbol{x},t\,|\,\ell) = \widetilde{U}(\boldsymbol{x},t\,|\,\ell) - \frac{1}{V'(\boldsymbol{x},\ell)}\int_{V'(\boldsymbol{x},\ell)} \widetilde{U}(\boldsymbol{x},t\,|\,\ell/\alpha)\,d\boldsymbol{x}' \tag{5}$$

where

$$\int_{V'(\boldsymbol{x},\ell)} ....\ \ d\boldsymbol{x}'$$

denotes the volume integral over $\boldsymbol{x}'$ such that

$$V(\boldsymbol{x}',\ell/\alpha) \subset V(\boldsymbol{x},\ell)\,.$$

Next, we introduce the temporal change of the internal energy U. Recall that it depends on the frame of reference in general. Therefore we have to choose the frame of reference appropriately. It may be natural to define the temporal change of the energy in the frame moving with the translational velocity (4). The velocity and the acceleration of fluid particles in this frame are $\boldsymbol{u}(\boldsymbol{x},t) - \left\langle \boldsymbol{u}(\boldsymbol{x},t)\right\rangle_\ell$ and $\boldsymbol{a}(\boldsymbol{x},t)$, then the energy gain (per unit time) of each fluid particle in this frame is

$$\left(\boldsymbol{u}(\boldsymbol{x},t) - \left\langle \boldsymbol{u}(\boldsymbol{x},t)\right\rangle_\ell\right)\cdot \boldsymbol{a}(\boldsymbol{x},t)\,.$$

Therefore, the total energy gain per unit time and mass in the cube $V(\boldsymbol{x},\ell)$ is

$$\widetilde{T}(\boldsymbol{x},t\,|\,\ell) = \frac{1}{\ell^3}\int_{V(\boldsymbol{x},\ell)} \left(\boldsymbol{u}(\boldsymbol{x},t) - \left\langle \boldsymbol{u}(\boldsymbol{x},t)\right\rangle_\ell\right)\cdot \boldsymbol{a}(\boldsymbol{x},t)\ d\boldsymbol{x}' \tag{6}$$

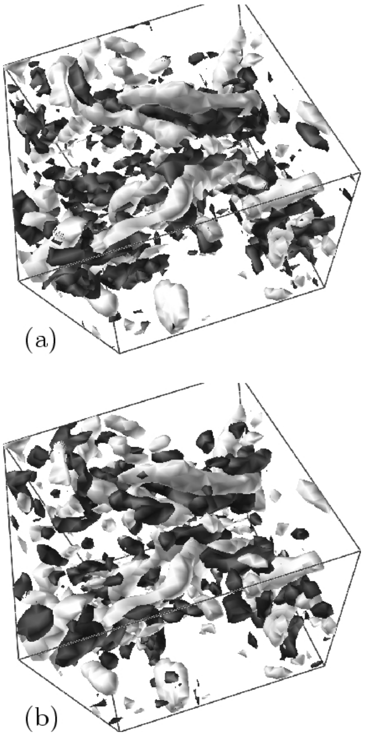

Fig. 2 (a) Iso-surfaces (light-colored objects) of the coarse-grained enstrophy and those (dark-colored objects) of the negative energy transfer. ℓ = 74η. (b) Iso-surfaces (light-colored objects) of the coarse-grained enstrophy and those (dark-colored objects) of the coarse-grained strain rate. ℓ = 74η. All the thresholds are taken as m + 1.5σ. The box size is $(3.5\ L)^3 \approx (600\eta)^3$

Then, the energy gain (hereafter which is called the energy transfer) of the structures between the scales between ℓ and ℓ/α is expressed as

$$T(\boldsymbol{x},t\,|\,\ell) = \widetilde{T}(\boldsymbol{x},t\,|\,\ell) - \frac{1}{V'(\boldsymbol{x},\ell)}\int_{V'(\boldsymbol{x},\ell)} \widetilde{T}(\boldsymbol{x},t\,|\,\ell/\alpha)\,d\boldsymbol{x}' \qquad (7)$$

It is emphasized again that the internal energy U and its transfer rate T are Galilean invariants.

3.3 Coherent structures relevant to the energy and its transfer

Iso-surfaces of the internal energy at ℓ = 18η and 74η are plotted in Figs. 1(a) and (c), respectively. It is clearly observed that these iso-surfaces are quite similar to those (Figs. 1b, d) of coarse-grained enstrophy at each scale, which is obtained by the low-pass filtering of the Fourier modes of velocity with the cut-off wave number $k_c = 2\pi/\ell$. Here, note that the thresholds are chosen by the mean values and standard deviations in a common manner. Note also that the vortical structures observed in Fig. 1(b, d) have tubular shapes with radii of $O(\ell)$. Hence, the coincidence of the two structures (of coarse-grained enstrophy and the internal energy) strongly suggests that the energy at scale ℓ is mainly possessed by vortex tubes with radii of $O(\ell)$. This implies that the energy cascade is the process in which larger (fatter) vortex tubes create smaller (thinner) vortex tubes.

Next, we plot in Fig. 2(a) the iso-surfaces (dark-colored objects) of negative energy transfer (at ℓ = 74η) together with the iso-surfaces (light-colored objects) of coarse-grained enstrophy. At this relatively large scale, the regions where the internal energy is lost (i.e. probably where the energy transfers to smaller scales) are located between the coherent vortices defined by the coarse-grained enstrophy. This is in contrast with the fact that the energy is confined in the vertical structures. For comparison, we plot in Fig. 2(b) the iso-surfaces (dark-colored objects) of the coarse-grained strain rate at the same scale (ℓ = 74η). It can be observed that the coarse-grained strain rate is also concentrated between the vortex tubes. The comparison of these two figures in Fig. 2 suggests that the negative energy transfer at a scale ℓ takes place in high coarse-grained strain regions (at the scale ℓ) which are located between the coherent vortex tubes at ℓ. This observation implies that the energy cascade may be due to the stretching of smaller-scale vortex tubes in the straining regions between larger-scale vortex tubes.

It is also interesting to observe that there are many anti-parallel pairs of vortex tubes in these figures. This is important because the (coarse-grained) strain rate becomes large around such pairs. Since vortex tubes at the Kolmogorov length[15] and vortex filaments[20] have the tendency of anti-parallel pairing, vortex tubes at larger scales might also tend to align to each other in an anti-parallel manner. Although we need to verify this feature more quantitatively, such anti-parallel pairing of vortex tubes is likely to play an important role in the energy cascade process because anti-parallel pairs of vortex tubes have ability to stretch vortices strongly in the perpendicular direction to them.

3.4 Examples supporting the scenario

It is shown numerically in the preceding subsection that the internal energy of fluid particles at a scale ℓ is possessed mainly by tubular vortices with radii of $O(\ell)$, and that the energy transfer takes place in straining regions around pairs of these vortex tubes. These results fairly support the scenario described in Section 2. In this subsection, in order to show examples further supporting the scenario, we investigate the regeneration process (Fig. 3) of small-scale structures in an artificial velocity field in which they are removed by the low-pass filtering of the Fourier modes of velocity at a moment of time. As seen in Fig. 3(a), only large-scale vortices (light-colored objects) exist at the initial time. However, as seen in Figs. 3(c–e), when a pair of such large-scale vortex tubes approach to each other (in an anti-parallel manner), smaller-scale vortex tubes (dark-colored objects) are created around the pair. It is also observed that the created vortex tubes tend to wrap, in the perpendicular direction, the parent vortex pair. Such creations of thinner vortex tubes are understood to be due to the vortex stretching in the strong straining (in the perpendicular direction) region which is created around the anti-parallel pair of fatter vortex tubes. This observation together with other similar examples (not shown here) also support the scenario suggested in Section 2.

4. CONCLUSION

In order to investigate the physical mechanism of the energy cascade process in turbulence, we have introduced the scale-dependent energy (5) and its transfer rate (7) as a function of position and time. The analyses of DNS data of statistically homogeneous isotropic turbulence reveal that the energy at each scale ℓ in the inertial range is mainly possessed

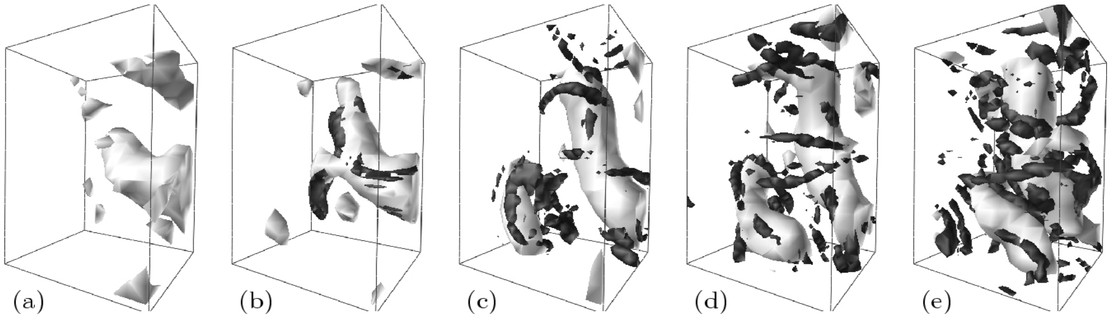

Fig. 3 Regeneration process of small-scale vortices. The size of shown box is $180\eta \times 180\eta \times 300\eta$. (a) Initially ($t = 0$), structures smaller than 37η are artificially removed by low-pass filtering of Fourier modes of the velocity. (b) $t = 6.6\tau_\eta$ (0.28T), (c) $t = 13\tau_\eta$ (0.56T), (d) $t = 20\tau_\eta$ (0.83T) and (e) $t = 27\tau_\eta$ (1.1T). Here, τ_η is the Kolmogorov time, and T is the integral time. When a pair of large-scale vortex tubes (light-colored objects, which are the iso-surfaces of enstrophy coarse-grained at 74η) approaches to each other, small-scale vortex tubes (dark-colored objects, which are the iso-surfaces of enstrophy coarse-grained at 18η) are created in the perpendicular direction to the parent vortex pair.

by tubular vortices with radii of $O(\ell)$, and that the energy transfer to smaller scales takes place around the pairs of these vortex tubes. We have also investigated the regeneration process of small-scale structures in an artificial velocity field, in which small-scale structures are removed initially. Then, it is frequently observed that when a pair of larger-scale vortices (i.e. fatter vortex tubes) align in an anti-parallel manner, smaller-scale vortices (i.e. thinner vortex tubes) are stretched and created in the strong straining region around the pair. It has been also confirmed (see Ref.[6] for more details) that the larger scales lose their energy in such straining regions, whereas smaller scales acquire the energy there. These DNS observations strongly support the scenario of the energy cascade described in Section 2.

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