Closed-form Capacity Analysis for MIMO Rayleigh Channels

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ABSTRACT

In this letter, we derive a tight closed form formula for an ergodic capacity of a multiple-input multiple-output (MIMO) for the application of wireless communications. The derived expression is a simple close-form formula to determine the ergodic capacity of MIMO systems. Assuming the channels are independent and identically distributed (i.i.d.) Rayleigh flat-fading between antenna pairs, the ergodic capacity can be expressed in a closed form as the finite sum of exponential integrals.

Key Words - Channel capacity, multiple-input multipe-output (MIMO), wireless communications.

Ⅰ. Introduction

The concepts of multiple-antenna have become the main concentration of wireless communications providing, high spectral efficiencies. Through an increased spatial dimension, the multiple-input multiple-output (MIMO) is providing a dramatic channel capacity gain. Due to an increased number of channel parameters in the receiver side it is very difficult to obtain perfect CSI. So, channel capacity with imperfect CSI is the main inspecting dilemma. Shannon's famous channel capacity formula introduced a new era for wireless communications. Considering different constraint as transmission matrix and constant power constraint. Telatar [1] derived the integral formula for ergodic channel capacity of independent and identically distributed (i.i.d) Rayleigh flat-fading MIMO channels. In MIMO systems multiple antennas at both transmit and receive ends have recently drawn significant concentration in response to the increasing requirements on data rate and quality in communication systems [2]-[6].

The main objective of this paper is to extend the analysis in [1] to obtain simple closed-form formulas for the ergodic capacity of i.i.d Rayleigh flat-fading MIMO channels. By analyzing the integral channel capacity formula provided in [1], H. Shin [5] extends the integral expression to a closed form solution. Gans [6] first evaluate the probability density function (pdf) of the received instantaneous SNR under channel estimation error. By using the pdf obtained by Gans [6], we extend the integral formula provided by Telatar [1] and obtained a closed form simple ergodic capacity formula. The derived expression for the ergodic channel capacity is simple and provides less complexity by replacing the infinite integration with finite summation.

The remainder of this paper is organized as follows. The next section provides channel model of MIMO systems. In section III, we derive the closed form formula for the MIMO ergodic capacity, followed by simulation result . Finally, section IV concludes the paper.

Ⅱ. Channel Model of OSTBC

For simplicity we consider a communication system of n_t transmit and n_r receive antennas. Information transformation is done through mapping the bits in a particular signal constellation. The corresponding mapped symbols are encoded by transmission matrix X

of an OSTBC; X_{ij} is the linear combination of the symbols and their complex conjugates which are transmitted through i^{th} transmit antenna in the j^{th} time slot. The rate of an OSTBC is defined as the ratio between the number of symbols the encoder takes as its input and the number of space time coded symbols transmitted from each antenna. So, the OSTBC rate is expressed as $R = k/N$, where k symbols are transmitted through N time slots. From [4], Transmission matrix of an OSTBC is related such that $XX^H = r\sum_{m=1}^{M}|s_m|^2 I_{n_t}$. The constant depends on the transmission matrix X.

The input-output relation of conventional MIMO systems can be written as

$$\boldsymbol{Y} = \boldsymbol{HX} + \boldsymbol{W} \tag{1}$$

where the received signal Y is an $n_r \times N$ matrix. H is an channel matrix. X is an $n_t \times N$ transmission matrix, and the noise W is an $n_r \times N$ matrix with i.i.d. complex circular Gaussian random variables, with zero mean and σ^2 variance. The MIMO channel model is described by a $n_r \times n_t$ matrix H, where h_{ij} is a complex Gaussian variable describing the channel from the j^{th} transmit antenna to the i^{th} receive antenna. It assumes that the channel matrix H is normalized in a way such that $E[\| \mathbf{H} \|_F^2] = n_r n_t = M$ .The average energy of the transmitted symbol from each antenna are assumed to be $\mathbf{E}/n_t$ so that the average power of the received signal at each received antenna is equal to **E** and the average SNR per received antenna is $\mathbf{E}/\sigma^2$. Maximum Likelihood (ML) decoder can be used at the receiver end to detect the transmitted symbol with significant accuracy. Using ML decoder at the receiver end, received signal can be expressed as

$$y = r\|H\|_F^2 x_n + w_k \tag{2}$$

where $r\|H\|_F^2 \sigma_k^2$ represent noise term with zero mean, and variance. In [4] the effective signal-to-noise ratio (SNR) per symbol is expressed as

$$\vartheta = \frac{\bar{\vartheta}}{n_t R}\|H\|_F^2 \tag{3}$$

where $\bar{\vartheta} = E_s/\sigma_k^2$ is the average SNR per receive antenna. Also in [4] the correlation coefficient is determined as $\rho = \sqrt{1-\varepsilon^2}$, where $\varepsilon \in [0,1]$ is the measure of the accuracy of the channel estimation. Under channel estimation error, the probability density function of the received instantaneous SNR, evaluated by Gans [6] is expressed as

$$\tau_\vartheta(\vartheta) = \frac{e^{-\frac{n_t R \vartheta}{\bar{\vartheta}(1-\rho^2)}}}{(1-\rho^2)^{1-M}} \frac{n_t R}{\bar{\vartheta}}\, {}_1F_1\left(M; 1; \frac{n_t R}{\bar{\vartheta}} \cdot \frac{\rho^2}{(1-\rho^2)^2}\right)$$

$$= \frac{e^{-\frac{n_t R \vartheta}{\bar{\vartheta}(1-\rho^2)}}}{(1-\rho^2)^{1-M}} \frac{n_t R}{\bar{\vartheta}} \sum_{k=0}^{M-1} \left(\frac{1}{k!}\right)\binom{M-1}{k}\left(\frac{n_t R}{\bar{\vartheta}} \cdot \frac{\rho^2}{(1-\rho^2)^2}\right)^k \tag{4}$$

where $M = n_t n_r$, and ${}_1F_1(a;b;z)$ is the confluent hypergeometric function of the first kind defined as ${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!}$, where $(a)_k$ and $(b)_k$ are Pochhammer symbol [7]. The Pochhammer symbol is defined as $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1)$ and $\binom{n}{k} = \left(\frac{n!}{k!\,(n-k)!}\right)$ is the binomial coefficient.

III. Ergodic Capacity Formulas in Closed-Form

Considering the channel model discussed in previous sections we drive a closed form channel capacity for MIMO systems.

Theorem: The ergodic capacity in bits/s/Hz of an i.i.d. Rayleigh fading MIMO channel with transmit and receive antennas is given by

$$C = \frac{QR}{\ln(2)} \frac{\rho^2 e^Q}{(1-\rho^2)^{1-M}} \sum_{k=1}^{N}\sum_{j=1}^{k+1} Q^{k-j}\Gamma(j-k-1, Q) \tag{5}$$

Where $Q = n_t R/\vartheta$ and equal power is allocated to each transmitted antenna.

Proof: Given the pdf of ϑ, the ergodic capacity of the equivalent STBC channel is

$$C = R\int_0^\infty \log_2(1+\vartheta)\, r_\lambda(\vartheta)\, d\vartheta \qquad (6)$$

Inserting equation (4) in the above equation we get

$$C = \frac{R}{\ln(2)} \frac{1}{(1-\rho^2)^{1-N}} \frac{n_t R}{\vartheta} \sum_{k=0}^{N-1} \left(\frac{1}{k!}\right) \binom{N-1}{k} \left(\frac{n_t R}{\vartheta} \cdot \frac{\rho^2}{(1-\rho^2)^2}\right)$$

$$\times \int_0^\infty \ln(1+\vartheta)\vartheta^k e^{-\frac{n_t R}{\vartheta}} d\vartheta \qquad (7)$$

to evaluate the integral of the above equation we use the result of [5]

$$I_i(\alpha) = \int_0^\infty \ln(1+x)\, x^{i-1} e^{-\alpha x} dx \quad \alpha > 0, i = 1,2,3,\cdots$$

$$= (i-1)!\, e^\alpha \sum_{l=0}^{n} \frac{\Gamma(i-l,\alpha)}{\alpha^i} \qquad (8)$$

the complementary incomplete gamma function is defined by [7]

$$\Gamma(a,y) = \int_0^\infty e^{-v} v^{a-1} dv \qquad (9)$$

Applying equation (5), the integral in equation (4) can be evaluated as

$$I = k!\, e^{\frac{n_t R}{\vartheta}} \sum_{j=1}^{k+1} \frac{\Gamma\left(j-k-1, \frac{n_t R}{\vartheta}\right)}{\frac{n_t R}{\vartheta}^j} \qquad (10)$$

Inserting the integral value of equation (10) in equation (7), we the closed form ergodic capacity expression as expressed in (5).

When CSI is perfectly known, channel capacity can be expressed as

$$C = \frac{QRe^Q}{\ln(2)} \sum_{k=1}^{N} \sum_{j=1}^{k+1} Q^{k-j}\Gamma(j-k-1,Q) \qquad (11)$$

Figure 1 shows simulation result of the derived capacity. Simulation is done for the different number of antennas in both transmitter and receiver side. Simulation result shows that the capacity increased linearly because of equal number of antennas are deployed in both the transmitter and receiver.

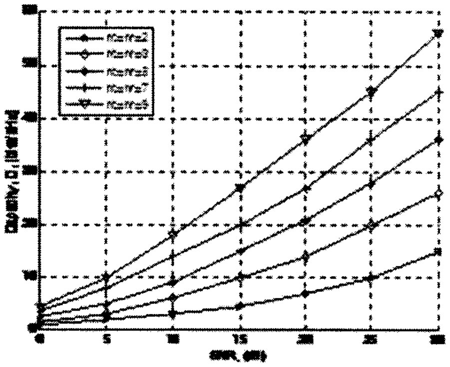

Figure 1. Closed-form ergodic channel capacity versus average signal to noise ratio. The legend denotes the number of transmit and receive antennas.

Ⅳ. Conclusion

In this paper, we derived a formula for the channel capacity, as a closed form solution, of MIMO wireless systems. Simulation was performed to get the closed-form ergodic channel capacity versus average signal to noise ratio. The derived capacity formula allows us to calculate the channel capacity in a closed form. Consequently, the derived formula can be applied for calculating the channel capacity of the practical MIMO system where a large number of antennas are used to transmit and receive data through wireless media.

Acknowledgement

This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (No. R11-2005-029-06003-0)

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