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An Improved Algorithm for a Capacitated Dynamic Lot-Sizing Problem with Two-Stage Supply Chain

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Abstract

This paper considers a two-stage dynamic lot-sizing problem constrained by a supplier's production capacity. We derive an improved O(T^6) algorithm over the O(T^7) algorithm in van Hoesel et al. (2005).

Keywords: Two-Stage Dynamic Lot-Sizing; Capacity; Inventory and Logistics; Algorithms

1. Introduction

In this paper, we consider a two-stage dynamic lot-sizing problem for a supply system between a supplier and a customer where the supplier's production capacity is limited. Considering the production capacity of the supplier, final goods are *produced* to the supplier's warehouse, out of which they are *replenished* to the customer's warehouse to ultimately fill up the customer's demands. To model the economies of scale in fulfilling demand, both of the production and replenishment cost functions are assumed to be concave.

Production and inventory planning for multi-stage supply chain was started by Zangwill (1969) with the assumption that production in every stage has no limitation on the production level. For the production capacitated case, Kaminsky and Simchi-Levi (2003) provided an $O(T^8)$ algorithm for a two-stage supply chain in which they treated the replenishment operation as transportation function. For the same problem, van Hoesel et al. (2005) developed an improved $O(T^7)$ algorithm. They pose the complexity reduction of their algorithm as an open question. In this paper, we address this question by deriving an $O(T^6)$ algorithm for solving the two-stage problem in the presence of production capacity constraint.

Section 2 defines the problem and provides optimality properties. Section 3 presents the optimal algorithm. Finally Section 4 gives concluding remarks and future research.

2. The Problem and Structural Properties

The production capacity is assumed to be normalized to one so that each demand size is scaled down in regard to the production capacity. Let T denote the length of planning horizon. For each period $t \in \{1, 2, \ldots, T\}$ we define:

- d_t : (scaled-down) demand in the customer's site;
- x_t: production quantity at the supplier's site in period t.;
- I_t^1 : inventory level at the supplier's warehouse in period t;
- $p_t^1(x_t)$: concave production cost function of the supplier for the amount x_t with $p_t^1(0) = 0$;
- $h_t^1(I_t^1)$: concave inventory holding cost in the supplier's warehouse for the amount I_t^1.
- y_t: replenishment quantity into the customer's warehouse in period t;
- $p_t^2(y_t)$: concave replenishment cost function of the customer for the amount y_t with $p_t^2(0) = 0$;
- I_t^2 : inventory level in the customer's warehouse;
- $h_t^2(I_t^2)$: concave inventory holding cost in the customer's warehouse for the amount I_t^2 .

For notational convenience, we let $v_{s,t} = v_s + v_{s+1} + \cdots + v_t$ if $s \le t$ and $v_{s,t} = 0$ if $s > t$, for any series of values $v_s, v_{s+1}, \ldots, v_t$. We use $\lceil x \rceil$ and $\lfloor x \rfloor$ to express the smallest integer no less than x and the largest integer no greater than x, respectively. The two-stage capacitated lot-sizing problem is formulated as:

$$\text{Min} \sum_{t=1}^{T}\left(p_t^1(x_t) + h_t^1(I_t^1) + p_t^2(y_t) + h_t^2(I_t^2)\right) \quad (1)$$

Subject to

$$I_{t-1}^1 + x_t = y_t + I_t^1, \qquad t = 1, \ldots, T, \quad (2)$$
$$I_{t-1}^2 + y_t = d_t + I_t^2, \qquad t = 1, \ldots, T, \quad (3)$$
$$x_t \le 1, \qquad t = 1, \ldots, T, \quad (4)$$
$$I_0^1 = I_T^1 = I_0^2 = I_T^2 = 0, \quad (5)$$
$$I_t^1 \ge 0,\ I_t^2 \ge 0, \qquad t = 1, \ldots, T. \quad (6)$$

Since the production capacity is stationary over period, it has been normalized to one as imposed in (4). A period t is called a *production period* (*replenishment period*) if $x_t^1 > 0$ ($y_t^1 > 0$), respectively. For a production period t, if it holds that $x_t = 1$, we call period t *FPL* (*full-production-capacity-load*); otherwise if $x_t < 1$, we call it *LPL* (*less-than-production-capacity-load*). Since we have no capacity limitation on replenishments in the customer side or the capacity is infinite, any replenishment period t is LPL.

For the efficient computation of inventory carrying cost, we define $h^2(Q|s, t)$ as the cost of carrying Q units during $[s, t]$ in the warehouse and in the center, respectively, i.e.,

$$h^2(Q|s,t) = \sum_{i=s}^{t} h_i^2(Q - d_{s+1,i})\,.$$

2.1 Minimum Concave Cost Network

We can view the problem as a minimum concave cost flow network problem. In the following we illustrate the network representation of a 12-period problem.

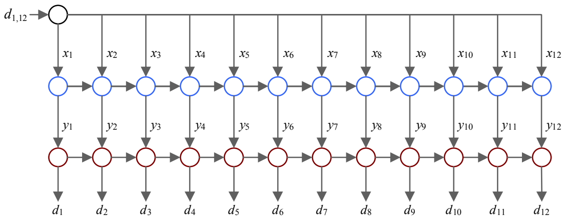

Figure 1. The network of production and replenishment with inventory.

The node having an entering arc with production quantity x_t is denoted by $(1, t)$ and that with replenishment y_t is denoted by $(2, t)$. Figure 2 illustrates a typical network flow corresponding to an extreme point solution of the problem (1)–(6).

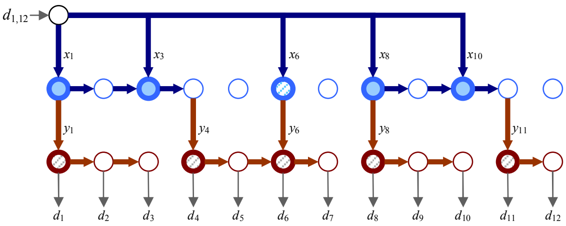

Figure 2. The subnetwork of non-zero flows.

In this figure, every node $(1, t)$ with no production or no replenishment in period t has no decoration. On the other hand, node $(1, t)$ is filled with color if the the production in period t is FPL, and filled with slanted stripes if it is LPL. Note that every replenishment is either zero or LPL so that any node $(2, t)$ with replenishment quantity greater than zero is filled with slanted stripes. Now we are interested in a *reduced subnetwork* of Figure 2 that contains only the flows corresponding to replenishment, and inventory levels in the supplier's and curstomer's warehouses as shown in Figure 3.

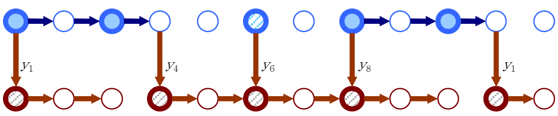

Figure 3. The reduced subnetwork of inventory and replenishment flows.

We say that any two nodes (i, s) and (j, t) are *connected* if there exists a path between them in the reduced subnetwork. We also say that $(1, s)$ and $(1, t)$ are *connected in the supplier's horizon* if $I_i^1 > 0$ for $i = s, s+1, ..., t-1$. Similary, nodes $(2, s)$ and $(2, t)$ are said to be *connected in the customer's horizon* if $I_i^2 > 0$ for $i = s, s+1, ..., t-1$. A set of connected nodes defines a *regeneration group*. We can represent the regeneration group using the first and latest supplier's connected nodes $(1, \alpha)$ and $(1, \beta)$ with the first and latest customer's connected nodes $(2, \lambda)$ and $(2, \gamma)$. Note that $\alpha \le \lambda \le \beta \le \gamma$. Since we do not allow backlogging, we have production and replenishment in periods α and λ, respectively. Moreover, the final replenishment in the regeneration group occurs in period β since $I_{\beta-1}^1 + x_\beta > 0$ and $I_\beta^1 = 0$. We denote each regeneration group by $(\alpha, \lambda, \beta, \gamma)$, for which it holds that

(i) $I_{\alpha-1}^1 = I_{\lambda-1}^2 = I_\beta^1 = I_\gamma^2 = 0,$
(ii) $I_i^1 > 0$ for each $i = \alpha, \alpha+1, \dots, \lambda-1,$
(iii) $I_i^2 > 0$ for each $i = \beta+1, \beta+2, \dots, \gamma.$

The network in Figure 3 shows a (1, 1, 11, 12) regeneration group.

Any flow between its lower and upper bound is called a *free flow* or *free arc*. It is well-known that the subnetwork involving only free arcs contains no cycle (Zangwill 1969; Ahuja et al. 1993). In our problem, note that each inventory level and each replenishment have no upper bound. So, we can see that each arc corresponding to carrying inventory and placing a replenishment is LPL by definition, which is free flow in the network. To characterize the properties in extreme point solutions, we are interested in the subnetwork only having LPL productions. The subnetwork of the free flows is the one from removing the arcs of FPL productions x_1, x_3, x_8 and x_{10} in Figure 2. We can derive the following from the no-cycle property, so-called the *fractional production principle*, which is shown in van Hoesal et al. (2005).

Proposition 1. In a regeneration group, there exists at most one LPL production period.

Hereafter we will derive properties by focusing on the reduced subnetwork in Figure 3. Applying the no-cycle property to the reduced subnetwork, we can show the following.

Proposition 2. For consecutive replenishment periods s and t in the same regeneration group, it holds that $I_s^1 \cdot I_{t-1}^2 = 0$ and $I_s^1 + I_{t-1}^2 > 0$.

2.2 Cumulative Production Quantity and Fractional Production Principle

Given a regeneration group $(\alpha, \lambda, \beta, \gamma)$, the production quantity in each period t is restriced to one of zero, fractional or at full capacity by the fractional production principle. We define Δ as the fractional quantity of the regeneration group: $\Delta \equiv d_{\lambda\gamma} - \lfloor d_{\lambda\gamma} \rfloor$. Hereafter, we use Δ instead of $d_{\lambda\gamma} - \lfloor d_{\lambda\gamma} \rfloor$ if there is no ambiguity. We use ι_s (= 0, Δ or 1), to describe the production quantity of period s; ι_s = 0 (Δ, 1) denotes zero (LPL, FPL) production in period s, respectively, The fractional production principle (Proposition 1) suggests that there is at most one period s with $\iota_s = \Delta$ during $[\alpha, \beta]$. Consider the total cumultative production quantity from a period s to the last period t, i.e., $\iota_s + \iota_{s+1} + \cdots + \iota_t$ ($\equiv \iota_{st}$). If the value ι_{st} is integer, then it means that ι_{st} FPL productions are scheduled to occur during $[s, t]$. However, if ι_{st} is fractional, then we know that $\iota_{st} - 1$ FPL productions and one LPL production are going to occur during $[s, t]$. Such value ι_{st} has the role of restricting the production quantity in period s–1. If the value ι_{st} is fractional, then the production quantity ι_{s-1} cannot be Δ but must be either 0 or 1. To ensure the fractional production principle, we devise an operator "⊕": for arbitrary two numbers a and b,

$$a \oplus b = \begin{cases} a+b, & \text{if either } a \text{ or } b \text{ is integer,} \\ \infty, & \text{otherwise.} \end{cases}$$

Therefore, given the total production quantity $\iota_{s\beta}$ since period s through the final period β in the supplier's horizon, the total production quantity from period s–1 is $\iota_{s-1} \oplus \iota_{s\beta}$, which will not be infinite provided that either ι_{s-1} or $\iota_{s\beta}$ is an integer. Thus the legitimacy of total quantity since period s–1 is assured by the operator "⊕". Finally, we use $\Omega(\lambda, \gamma)$ to denote the set of possible cumulative production quantities in regeneration group $(\alpha, \lambda, \beta, \gamma)$; $\Omega(\lambda, \gamma) = \{0, 1, 2, \ldots, \lfloor d_{\lambda\gamma} \rfloor\} \cup \{\Delta, 1+\Delta, \ldots, \lfloor d_{\lambda\gamma} \rfloor + \Delta\}$

2.3 Determination of Replenishment Quantities and Costs

Production and replenishment plan is obtained once production quantities $x_1, \ldots, x_T$ and replenishment quantities $y_1, \ldots, y_T$ are determined. Single-stage lot-sizing problems in the literature have been solved by determining production quantities applying ZIO (*zero-inventory ordering*) principle in uncapacitated cases (Wagner and Whitin 1957; Zangwill 1969) and applying fractional production principle in capacitated cases (Florian and Klein 1973). Since our problem has capacity constraint in the supplier, the production quantities, in a given regeneration group, can be determined using the fractional production principle. In the same way, one might expect that the replenishment quantities can be obtained using the ZIO principle because the customer's replenishing operations are uncapacitated. However, a simple application of ZIO principle cannot yield an optimal solution because production quantities affect the replenishment quantities. Therefore, a replenishment quantity, in most cases, cannot be expressed as a sum of a series of demands. As we shall see later, a replenishment quantity depends on the cumulative production quantity and the structure between replenishment periods. To understand their structure, we observe replenishment types and patterns between them in the following.

2.3.1 Replenishment Types and Patterns

We will classify replenishment periods t by the inventory level I_{t-1}^2 at the beginning of t in the customer's warehouse. Any replenishment period t with $I_{t-1}^2 > 0$ is called a *transfer period* (also called *type-F* period) and that with $I_{t-1}^2 = 0$ is called a *terminal period* (also called *type-E* preiod). As a corollarly of Proposition 2, we have the following property.

Proposition 3. A transfer replenishment period in t implies another replenishment period $s < t$ with $I_s^1 = 0$.

Consider an extreme solution (x, y) that has a regeneration group $(\alpha, \lambda, \beta, \gamma)$ with consecutive replenishment periods s and t, $\lambda \le s < t \le \beta$. We address four sorts of replenishment patterns between replenishment periods s and t. Pattern *EE* describes the situation that both s and t are terminal periods. Since nodes (2, s) and (2, t) are not connected in customer's horizon in the network corresponding to the solution (x, y), nodes (1, s) and (1, t) should be connected in supplier's horizon since periods s and t belong to the same regeneration group. In pattern *EF* specifying that period s is terminal and period t is transfer, nodes (2, s) and (2, t) are connected in the customer's horizon and nodes (1, s) and (2, t) should be disconnected in the supplier's horizon. Furthermore, Proposition 3 for this case suggests that $I_s^1 = 0$. We next consider the *FE* in which s and t are transfer and terminal periods, respectively. In this case, we note that nodes (2, s) and (2, t) are not connected in the customer's horizon. Finally, in the *FF* pattern, both s and t are transfer periods where nodes (1, s) and (1, t) are disconnected in the supplier's horizon. Figure 4 illustrates the four patterns in replenishments.

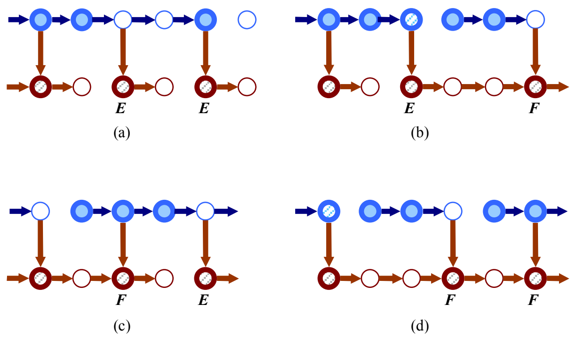

Figure 4. Replenishment Patterns

2.4 Replenishment Quantities and Costs

As stated earlier, determination of replenishment quantities requires the scheduled cumulative production quantity and replenishment types and patterns. Given a replenishment period s, we are interested in the total cumulative production quantity after s thorugh the final period β in the supplier's horizon. We let $n_s = \iota_{s+1,\beta}$ to denote the total cumulative production quantity after period s. If period s is transfer, it suggests its previous replenishment period, say, s', from Proposition 3. In general, we let $m_s = \iota_{s'+1,s}$ to denote the cumulative production quantity after its previous replenishment period through period s. Now we provide replenishment quantities in period s for the four patterns between periods s and t, $\lambda \le s \le t$–1 $\le \gamma$ and the costs in satisfying demands $d_s, d_{s+1}, \ldots, d_{t-1}$. It should be noted that period s is a replenishment period whereas period t might not be a replenishment period. Consider the *EE* situation of $I_{s-1}^2 = 0$ and $I_{t-1}^2 = 0$. In this case, the well-known ZIO principle directly applies. That is, the replenishment quantity of s is given as $y_s = d_{st-1}$. The d_{st-1}

units are used to satisfy demands $d_s, d_{s+1}, \ldots, d_{t-1}$ with replenishment and inventory holding costs in the customer's warehouse given as

$$p_s^2(d_{s,t-1}) + h^2(d_{s+1,t-1}|s, t-1). \quad (7)$$

Under *EF* situation, since $I_s^1 = 0$ (from $I_{t-1}^2 > 0$ and Proposition 3), all the units replenished in period s should be allocated to demands $d_s, d_{s+1}, ..., d_\gamma$. Noting that $I_{s-1}^2 = 0$ and that the total cumulative production quantity after period s in the regeneration group is given by n_s, we have $y_s + n_s = d_{s\gamma}$, i.e., $y_s = d_{s\gamma} - n_s$, for which the replenishment and holding costs for demands $d_s, d_{s+1}, \ldots, d_{t-1}$ are given as

$$p_s^2(d_{s\gamma} - n_s) + h^2(d_{s+1,\gamma} - n_s|s, t-1). \quad (8)$$

Conider *FE* situation. Since period s is a transfer period, it has its previous replenishment period. Let the scheduled production quantity between them be m_s. Then the total replenishment quantity since the previous period is $m_s + n_s$. Among the total $m_s + n_s$ units, we see that $m_s + n_s - d_{t\gamma}$ units are allocated for demands $d_s, d_{s+1}, ..., d_{t-1}$ since $I_{t-1}^2 = 0$. Hence, the replenishment in period s should be the one remaining after satisfying $d_{t\gamma}$ units. Thus, we obtain $y_s = m + n - d_{t\gamma}$. Since period t is Type-E ($I_{t-1}^2 = 0$), exactly $d_{s,t-1}$ units should be dispatched for demands $d_s, d_{s+1}, ..., d_{t-1}$, incurring holding cost of $h^2(d_{s+1,t-1}|s, t-1)$. Hence, the total cost of replenishment and carrying inventory during $[s, t-1]$ is

$$p_s^2(m_s + n_s - d_{t\gamma}) + h^2(d_{s+1,t-1}|s, t-1) \quad (9)$$

We finally deal with *FF* situation. In this case, the units produced from the previous production period of s through period s are all replenished in period s. That is, $y_s = m_s$. Using similar arguments in (8), we also see that the holding cost for $d_s, d_{s+1}, ..., d_{t-1}$ is $h^2(d_{s+1,\gamma} - n_s|s, t-1)$. Thus, the total cost of replenishment and carrying inventory is

$$p_s^2(m_s) + h^2(d_{s+1,\gamma} - n_s|s, t-1). \quad (10)$$

3. Optimal Dynamic Programming Algorithm

3.1 Definitions

The solution approach is basically to decompose the problem (1)–(6) by regeneration groups $(\alpha, \lambda, \beta, \gamma)$. Note that the production during $[\alpha, \lambda]$ is all supplied for demands $d_\lambda, d_{\lambda+1}, ..., d_\gamma$ and no replenishment takes place during $[\beta+1, \gamma]$. With the respect to the regeneration group $(\alpha, \lambda, \beta, \gamma)$, we consider three intervals $[1, \lambda-1]$, $[1, \beta-1]$ and $[1, \gamma]$. The problem for satisfying the demands during $[1, \gamma]$ will be handled by procedure $F(\cdot)$ and that for demands during $[1, \beta-1]$ by procedures $f(\cdot)$ and $\pi(\cdot)$. Those procedures $F(\cdot)$, $f(\cdot)$ and $\pi(\cdot)$, for the intervals $[1, \beta-1]$ and $[1, \gamma]$, will mainly focus on how to replenish demands in the regeneration group. However, the probelm related with interval $[1, \lambda-1]$, which will be solved by $G(\cdot)$, consists of making two schedules: one is a schedule of purely productions (without replenishments) during $[\alpha, \lambda]$ for demands in the regeneration group $(\alpha, \lambda, \beta, \gamma)$ and the other one is a schedule of replenishments (with productions) during $[1, \lambda-1]$.

Note that given two consecutive regeneration groups $(\alpha', \lambda', \beta', \lambda-1)$ and $(\alpha, \lambda, \beta, \gamma)$, we may possibly have a non-connected nodes $(1, \beta'+1), (1, \beta'+2), ..., (1, \beta-1)$, which do not belong to any regeneration group. To characterize these periods $\beta'+1, \beta'+2, ..., \beta-1$, we generalize the cost associated with productions during $[\alpha, \lambda]$. Let $G(m_s|s, \lambda, \gamma)$ be the minimum cost when satisfying demands $d_1, d_2, ..., d_{\lambda-1}$ and producing m_s units during $[1, s]$ under the assumption that λ is the first replenishment period of a regeneration group and no replenishment has occurred during $[s+1, \lambda-1]$.

We let $F(\beta, \gamma)$ be the minimum cost when satisfying demands $d_1, d_2, ..., d_\gamma$ under the constraint that β and γ are the latest periods in supplier's and customer's horizon in a regeneration group. So, we have a replenishment in period β but no replenishments during $[\beta+1, \gamma]$ when computing $F(\beta, \gamma)$. To obtain each value $F(\beta, \gamma)$, we need to identify a regeneration group $(\alpha, \lambda, \beta, \gamma)$ to which the two periods β and γ belong and to find the cost associated with the body $[\lambda, \beta]$ of the group. The cost of a body of a group will be computed based on replenishment types of period β.

Let $\pi_E(n|\lambda, s, t, \gamma)$, $\lambda \le s \le t-1 \le \gamma$, be the minimum cost in satisfying demands $d_1, d_2, \ldots, d_{t-1}$ under the constraints that

(i) periods λ and γ are the first and the last periods in customer's horizon in the same regeneration group,
(ii) the period t is type E,
(iii) replenishment occurs in period s and the total cumulative production quantity in the regeneration group after period s is n_s.

We also define $\pi_F(m_s, n_s|\lambda, s, t, \gamma)$ to be the same cost as $\pi_E(n_s|\lambda, s, t, \gamma)$ except that period t is type-F and the total production quantity scheduled during $[s'+1, s]$ is m_s where the latest replenishment period during $[1, s]$ is s'. The computations of $\pi_E(\cdot)$ and $\pi_F(\cdot)$ will require cost terms in association with replenishment patterns. Note that we have four replenishment patterns *EE*, *EF*, *FE* and *FF*.

Let $f_{XY}(n_s|\lambda, s, t, \gamma)$, $\lambda \le s < t \le \gamma$, $X, Y \in \{E, F\}$, be the minimum cost in satisfying demands $d_1, d_2, \ldots, d_{t-1}$ under the constraints that

(i) periods λ and γ are the first and the last periods in customer's horizon in the same regeneration group,
(ii) periods s and t are type-X and type-Y replenishment periods,
(iii) the total cumulative production quantity in the regeneration group after period s is n_s.

Now we put dynamic programming formulas to relate the costs $F(\cdot)$, $G(\cdot)$, π_X and $f_{XY}(\cdot)$. We start with computing $F(\cdot)$ and $G(\cdot)$.

3.2 Main Dynamic Programming Procedures

We first consider how to compute $F(\beta, \gamma)$. Suppose that periods β and γ are the latest two periods in supplier's and customer's horizons, respectively, say, in a regenearion group (α, λ, β, γ). Note that $I_\gamma^2 = 0$, implying that period γ+1 is Type-E. We describe the cost $F(\beta, \gamma)$ based on the type of period β. Suppose that it is Type-E. Then we have EE pattern between period β and γ+1, which gives $y_\beta = d_{\beta\gamma}$ being supplied for demands $d_\beta, d_{\beta+1}, ..., d_\gamma$ with cost $p_\beta^2(d_{\beta\gamma}) + h^2(d_{\beta+1,\gamma}|\beta, \gamma-1)$ as in (7). Now consider the cost for supplying demands $d_1, d_2, ..., d_{\beta-1}$. Since period β is Type-E and the total production quantity after period β is zero, we know that the cost for $d_1, d_2, ..., d_{\beta-1}$ is $\pi_E(0|\lambda, \beta, \beta, \gamma)$. Hence, we have

$$F(\beta, \gamma) = \pi_E(0|\lambda, \beta, \beta, \gamma) + p_\beta^2(d_{\beta\gamma}) + h^2(d_{\beta+1,\gamma}|\beta, \gamma).$$

On the other hand if period β is Type-F, we have FE pattern between period β and γ+1, which yields $y_\beta = m_\beta + n_\beta - d_{\gamma+1,\gamma}$ where m_β is the cumulative production quantity scheduled after the previous period of β and $n_\beta = 0$. We note that the total cost for demands $d_\beta, d_{\beta+1}, ..., d_\gamma$ is $p_\beta^1(m_\beta) + h^1(d_{\beta+1,\gamma}|\beta, \gamma)$ by (8). Now consider the cost for the demands $d_1, d_2, ..., d_{\beta-1}$. Since no unit is produced from β to period γ and m_β units should be produced from period β back to its previous replenishment period, we can see that it takes cost of $\pi_F(m_\beta, 0|\lambda, \beta, \beta, \gamma)$ to satisfy demands $d_1, d_2, ..., d_{\beta-1}$. We thus obtain

$$F(\beta, \gamma) = \pi_F(m_\beta, 0|\lambda, \beta, \beta, \gamma) + p_\beta^2(m_\beta) + h^2(d_{\beta+1,\gamma}|\beta, \gamma).$$

Combininng the formulas above, we have a complete formula for $F(\beta, \gamma)$:

$$F(\beta,\gamma) = \begin{cases} \pi_E(0 \mid \lambda,\beta,\beta,\gamma) + p_\beta^2(d_{\beta\gamma}) + h^2(d_{\beta+1,\gamma} \mid \beta,\gamma), \\ \pi_F(m_\beta, 0 \mid \lambda,\beta,\beta,\gamma) + p_\beta^2(m_\beta) + h^2(d_{\beta+1,\gamma} \mid \beta,\gamma) \\ \quad : m_\beta \in \Omega(\lambda,\gamma). \end{cases}$$

Note that we can compute the cost $F(\beta, \gamma)$ in $O(T^2)$ if necessary values of $\pi_E(\cdot)$ and $\pi_F(\cdot)$ are preprocessed. Hence, every $F(\beta, \gamma)$ can be obtained in $O(T^4)$. So, the optimal cost is $F(T, T)$ is given in $O(T^4)$ once necessary values are preprocessed.

Now we explain how to compute $G(m_s|s, \lambda, \gamma)$. Suppose that period s is a replenishment period. Then, the units produced at or before period s are not allocated for demands $d_\lambda, d_{\lambda+1}, \ldots, d_T$, since period λ does not belong to the regeneration group containing period s. Note that this case is valid only when $m_s = 0$. We begin a new regeneration group of period whose latest replenishment period is s and last period is λ–1. Thus we have the cost of $F(s, \lambda-1)$ for demands $d_1, d_2, \ldots, d_{\lambda-1}$:

$$G(0|s, \lambda, \gamma) = F(s, \lambda-1).$$

Now suppose that we have no replenishment in period s. Let ι_s be the production quantity of period s for $\iota_s = 0, \Delta$ or 1 with production cost $p_s^1(\iota_s)$. Then the number of units to be produced before period s, all of which will be replenished in period λ, is $m_s - \iota_s$. With this information, the cost for demands $d_1, d_2, \ldots, d_{\lambda-1}$ is given by $G(m_s - \iota_s|s-1, \lambda, \gamma)$. Since we need to reserve m_s units at the end of period s, the corresponding holding cost is $h_s^1(m_s)$. So, the cost $G(m_s|s, \lambda, \gamma)$ in this case is obtained as

$$G(m_s|s, \lambda, \gamma) = G(m_s - \iota_s|s-1, \lambda, \gamma) + p_s^1(\iota_s) + h_s^1(m_s).$$

Combining the two formulas developed, for $m_s \in \Omega(\lambda, \gamma)$, $1 \le s \le \lambda \le \gamma$, we have

$$G(m_s \mid s,\lambda,\gamma) = \min \begin{cases} F(s, \lambda-1) : m_s = 0, \\ G(m_s - \iota_s \mid s-1,\lambda,\gamma) + p_s^1(\iota_s) + h_s^1(m_s) : \\ \quad \iota_s \in \{0,\Delta,1\},\ m_s - \iota_s \in \Omega(\lambda,\gamma). \end{cases}$$

With this formula, we can see that every $G(m_s|s, \lambda, \gamma)$ is obtained in $O(T^4)$.

3.3 Dynamic Procedures for Replenishment Types

3.3.1 Computing $\pi_E(n_s|\lambda, s, t, \gamma)$

Consider first the case that $s = \lambda$. In this case, we have no replenishment any longer before λ for the demands $d_\lambda, d_{\lambda+1}, \ldots, d_\gamma$ but only some productions, which should be replenished only at the period λ. Note that and period t is Type-E. Since perod λ is also Type-E ($I_{\lambda-1}^1 = 0$), the replenishment quantity y_λ in period λ is $d_{\lambda,t-1}$ and the total replenishment and holding cost to fulfill demands $d_{\lambda+1}, \ldots, d_{t-1}$ is $p_\lambda^2(d_{\lambda,t-1}) + h^2(d_{\lambda+1,t-1}|\lambda, t-1)$ from (7). Recall n_λ units have been produced during periods $[\lambda+1, \gamma]$, implying that the remaining unsatisfied $d_{\lambda\gamma} - n_\lambda$ units should be produced at or before period λ, which incurs cost of $G(d_{\lambda\gamma} - n_\lambda|\lambda, \lambda, \gamma)$. Hence, the cost $\pi_E(n_\lambda|\lambda, \lambda, t, \gamma)$ is given as

$$\pi_E(n_\lambda|\lambda, \lambda, t, \gamma) = G(d_{\lambda\gamma} - n_\lambda|\lambda, \lambda, \gamma) + p_\lambda^2(d_{\lambda,t-1}) + h^2(d_{\lambda+1,t-1}|\lambda, t-1).$$

We next consider the case where $s > \lambda$. In this case, we consider a subcase where period s is a replenishment period. If it is Type-E, then the cost $\pi_E(n_s|\lambda, s, t, \gamma)$ is given by $f_{EE}(n_s|\lambda, s, t, \gamma)$; otherwise if it is Type-F, then $\pi_E(n_s|\lambda, s, t, \gamma)$ equals to $f_{FE}(n_s|\lambda, s, t, \gamma)$. Thus we have

$$\pi_E(n_s|\lambda, s, t, \gamma) = \min\{f_{EE}(n_s|\lambda, s, t, \gamma), f_{FE}(n_s|\lambda, s, t, \gamma)\}.$$

We finally consider the other subcase that s is not a replenishment period where $\lambda < s \le \gamma$. Suppose that we have ι_s units produced in period s for $\iota_s = 0, \Delta$ or 1 for which production cost $p_s^1(\iota_s)$ is incurred. Then the total units produced after period $s-1$ is $n_s\square\iota_s \in \Omega(\lambda, \gamma)$. Note that $d_{t\gamma}$ units should be available at the beginning of period t while only n_s units have been produced during $[s+1, \gamma]$. Hence, we have to get $d_{t\gamma} - n_s$ units ready at the beginning of period s ($I_{s-1}^1 + \iota_s = d_{t\gamma} - n_s$), for which holding cost of $h_s^1(d_{t\gamma} - n_s)$ occurs in period s. So the cost for demands $d_1, d_2, \ldots, d_{t-1}$ with this information is given as

$$\pi_E(n_s|\lambda, s, t, \gamma) = \pi_F(n_s\square\iota_s|\lambda, s-1, t, \gamma) + p_s^1(\iota_s) + h_s^1(d_{t\gamma} - n_s).$$

The three formulas developed above provides a complete formula $\pi_E(n_s|\lambda, s, t, \gamma)$ as follows:

$$\pi_E(n_s \mid \lambda,s,t,\gamma)$$
$$=\begin{cases} G(d_{\lambda\gamma}-n_\lambda \mid \lambda,\lambda,\gamma)+p_\lambda^2(d_{\lambda,t-1})+h^2(d_{\lambda+1,t-1} \mid \lambda,t-1): \\ \quad \lambda = s,\ d_{\lambda\gamma}-n_\lambda \in \Omega(\lambda,\gamma), \\ \min\{f_{EE}(n_s \mid \lambda,s,t,\gamma), f_{FE}(n_s \mid \lambda,s,t,\gamma)\} : \lambda < s, \\ \pi_F(n_s \oplus \iota_s \mid \lambda,s-1,t,\gamma)+p_s^1(\iota_s)+h^2(d_{t\gamma}-n_s \mid s,t-1): \\ \quad \lambda < s,\ \iota_s \in \{0,\Delta,1\},\ n_s \oplus \iota_s \in \Omega(\lambda,\gamma). \end{cases}$$

3.3.2 Computing $\pi_F(m_s, n_s|\lambda, s, t, \gamma)$

Because period t is Type-F, it implies a replenishment period before t, say t'. The period t' may be period s or may not be. We recall that the quantity m_s represent the units to be produced during $[t'+1, s]$ and the quantity n_s the units that have been produced during $[s+1, \gamma]$. We explain the cost $\pi_F(m_s, n_s|\lambda, s, t, \gamma)$ when period s is λ. Because period λ is a replenishment period ($t' = \lambda$), in this case, the cost $\pi_F(m_\lambda, n_\lambda|\lambda, \lambda, t, \gamma)$ is valid only when $m_\lambda = 0$. Since perod λ is Type-E ($I_{\lambda-1}^1 = 0$) and period t is Type-F, the replenishment quantity is $d_{\lambda\gamma} - n_\lambda$, which incurs production and holding costs of $p_\lambda^2(d_{\lambda\gamma} - n_\lambda) + h^2(d_{\lambda+1,\gamma} - n_\lambda|\lambda, t-1)$ by (8). Note that the cost of satisfying demands $d_1, d_2, \ldots, d_{\lambda-1}$ and producing $d_{\lambda\gamma} - n_\lambda$ units during $[1, \lambda]$ is $G(d_{\lambda\gamma} - n_\lambda|\lambda, \lambda, \gamma)$. Thus the cost of $\pi_F(0, n_\lambda|\lambda, \lambda, t, \gamma)$ is given as

$$\pi_F(0, n_\lambda|\lambda, \lambda, t, \gamma) = G(d_{\lambda\gamma} - n_\lambda|\lambda, \lambda, \gamma) + p_\lambda^2(d_{\lambda\gamma} - n_\lambda) + h^2(d_{\lambda+1,\gamma} - n_\lambda|\lambda, t-1).$$

Now we consider the case when $s > \lambda$. We suppose that period s is a replenishment period. This case is valid only when $m_s = 0$ as described previously. If period s is Type-E, the cost $\pi_F(0, n_s|\lambda, s, t, \gamma)$ equals to $f_{EF}(n_s|\lambda, s, t, \gamma)$; otherwise, if period s is Type-F, the cost is given by $f_{FF}(n_s|\lambda, s, t, \gamma)$. Hence, we obtain

$$\pi_F(0, n_s|\lambda, s, t, \gamma) = \min\{f_{EF}(n_s|\lambda, s, t, \gamma), f_{FF}(n_s|\lambda, s, t, \gamma)\}.$$

Next suppose that s is not a replenishment period, $\lambda < s \le \gamma$. Suppose that we have ι_s (= 0, Δ, or 1) units produced in period s. Then the units to be produced at or before $s-1$ is $m_s - \iota_s$ all of which will be replenished in period t and the total produced quantity after period $s-1$ is $n_s \square \iota_s$. Note that we need to carry m_s units at the end of period s ($I_{s-1}^1 + \iota_s = m_s$) with holding cost $h_s^1(m_s)$. Hence

$$\pi_F(m_s, n_s|\lambda, s, t, \gamma) = \min\{\pi_F(m_s \exists \iota_s, n_s \square \iota_s|\lambda, s-1, t, \gamma) + p_s^1(\iota_s) + h_s^1(m_s)\}.$$

Combining the three formulas developed so far, we obtain the follwoing:

$$\pi_F(m_s,n_s \mid \lambda,s,t,\gamma)$$
$$=\begin{cases} G(d_{\lambda\gamma}-n_\lambda \mid \lambda,\lambda,\gamma)+p_\lambda^2(d_{\lambda\gamma}-n_\lambda)+h^2(d_{\lambda+1,\gamma}-n_\lambda \mid \lambda,t-1) \\ \quad : \lambda = s,\ m_\lambda = 0,\ d_{\lambda\gamma}-n_\lambda \in \Omega(\lambda,\gamma), \\ \min\{f_{EF}(n_s \mid \lambda,s,t,\gamma), f_{FF}(n_s \mid \lambda,s,t,\gamma)\} : \lambda < s,\ m_s = 0, \\ \pi_F(m_s-\iota_s, n_s \oplus \iota_s \mid \lambda,s-1,t,\gamma)+p_s^1(\iota_s)+h^1(m_s) \\ \quad : \lambda < s,\ \iota_s \in \{0,\Delta,1\},\ m_s-\iota_s \in \Omega(\lambda,\gamma),\ n_s \oplus \iota_s \in \Omega(\lambda,\gamma). \end{cases}$$

Note that the costs $\pi_E(n_s|\lambda, s, t, \gamma)$ and $\pi_F(m_s, n_s|\lambda, s, t, \gamma)$ can be obtained immediately if we are given necessary values of $f_{XY}(n_s|\lambda, s, t, \gamma)$ for all $X, Y \in \{E, F\}$. We also note that the size of the set $\Omega(\lambda, \gamma)$ is $O(T)$. Thus every $\pi_E(n_s|\lambda, s, t, \gamma)$ and $\pi_F(m_s, n_s|\lambda, s, t, \gamma)$ can be found in $O(T^6)$.

3.4 Dynamic Procedures for Replenishment Patterns

We first present a recusrion formula for the cost $f_{EE}(n_s|\lambda, s, t, \gamma)$. Recall that both periods s and t are replenishment periods of the same Type-E. Hence, the sum of replenishment cost for $d_{s,t-1}$ units and the inventory carrying cost during $[s+1, t-1]$ for demands $d_{s+1}, d_{s+2}, \ldots, d_{t-1}$ is $p_s^2(d_{s,t-1}) + h^2(d_{s+1,t-1}|s, t-1)$ as described in (7). With the knowledge of period s of Type-E and n_s units having been produced during $[s+1, \gamma]$, we can see that the cost for demands $d_1, d_2, \ldots, d_{s-1}$ is $\pi_E(n_s|\lambda, s, s, \gamma)$. The cost $f_{EE}(n_s|\lambda, s, t, \gamma)$ is then given by

$$f_{EE}(n_s|\lambda, s, t, \gamma) = \pi_E(n_s|\lambda, s, s, \gamma) + p_s^2(d_{s,t-1}) + h^2(d_{s+1,t-1}|s, t-1).$$

The formula for the cost $f_{EF}(n_s|\lambda, s, t, \gamma)$ can be easily obtained from (8) together with the formula of $f_{EE}(n_s|\lambda, s, t, \gamma)$:

$$f_{EF}(n_s|\lambda, s, t, \gamma) = \pi_E(n_s|\lambda, s, s, \gamma) + p_s^2(d_{s,\gamma} - n_s) + h^2(d_{s+1,\gamma} - n_s|s, t-1).$$

Now we deal with the formula for $f_{FE}(n_s|\lambda, s, t, \gamma)$. Since period s is Type-F, it has its previous replenishment peirod, say, s'. Suppose that we are going to produce m_s units during $[s'+1, s]$. Then from (9), the associated total sum of the replenishment cost in period s and the holding cost for demands $d_{s+1}, \ldots, d_{t-1}$ is $p_s^2(m_s + n_s - d_{t\gamma}) + h^2(d_{s+1,t-1}|s, t-1)$. With the m_s and n_s units about the period s, we can also see that the cost for demands $d_1, d_2, \ldots, d_{s-1}$ is $\pi_E(m_s, n_s|\lambda, s, s, \gamma)$. The cost $f_{FE}(n_s|\lambda, s, t, \gamma)$ is then obtained by the following equation:

$$f_{FE}(n_s|\lambda, s, t, \gamma) = \min\{\pi_F(m_s, n_s|\lambda, s, s, \gamma) + p_s^2(m_s + n_s - d_{t\gamma}) + h^2(d_{s+1,t-1}|s, t-1) : m_s \in \Omega(\lambda, \gamma), m_s \square n_s \in \Omega(\lambda, \gamma)\}.$$

Using similar reasoning in deriving the formula of $f_{FE}(n_s|\lambda, s, t, \gamma)$ and in (10), we can obtain the recursion formula of $f_{FF}(n_s|\lambda, s, t, \gamma)$ as follows:

$$f_{FF}(n_s|\lambda, s, t, \gamma) = \min\{\pi_F(m_s, n_s|\lambda, s, s, \gamma) + p_s^2(m_s) + h^2(d_{s+1,\gamma} - n_s|s, t-1) : m_s \in \Omega(\lambda, \gamma), m_s \square n_s \in \Omega(\lambda, \gamma)\}.$$

We can see that cumputation of $f_{XY}(n_s|\lambda, s, t, \gamma)$ takes at most $O(T^5)$ if the necessary values of $\pi_X(\cdot)$ are preprocessed for all $X, Y \in \{E, F\}$.

Thus all the values of $F(\cdot)$, $f(\cdot)$, $\pi(\cdot)$, and $G(\cdot)$ are computed in at most $O(T^6)$ and hence the optimal solution $F(T, T)$ is found in $O(T^6)$.

4. Concluding Remarks

In this paper, we considered a two-stage supply chain for production and replenishment schedules for a supplier and a customer. The chief achivement of the paper is the $O(T^6)$ optimal algorithm, which is an improvement on the $O(T^7)$ algorithm by van Hoesal et al. (2005). To complete the open questions addressed by them, more efficient solution procedures than $O(T^5)$ and $O(T^4)$ algorithms (van Hoesal et al. 2005) should be provided for nonspeculative replenishment costs and linear replenishment costs, respectively, which is under active study. The results in this paper may be helpful in extending the problem to the multi-statge supply chain enironments and to the case where customer's replenishment operations are capacitated.

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