The Application of Multigrid Algorithm to Low-Speed Precondition

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Abstract

The low-speed preconditioning technique is applied to solve the compressible Reynolds averaged Navier–Stokes equations for low-speed flows. The space discretization is based on Roe's flux-difference splitting with third-order-accurate MUSCL extrapolation. Time integration is performed employing a diagonal approximate factorization algorithm. The dual-time stepping has been incorporated to solve the unsteady flows. Full multigrid method is implemented to accelerate the convergence rate. To verify the algorithms several cases have been tested. Demonstrated the improvement on convergence and quality of the solution.

Introduction

Density based time-marching algorithms are widely used to solve the compressible Navier–Stokes equations. They have favorable convergence rate and accuracy in the computation of compressible flows. However, these algorithms cannot be directly applied to incompressible flows computation because of the "stiffness" of the governing equations. Sometimes there are problems containing mixed incompressible and compressible flows, and we also expect a single CFD solver can be used to compute flow fields from low-speed to high-speed flows. The preconditioning technique modifies the time-derivative term of N-S equations by pre-multiplying it with a preconditioning matrix, which rescale the eigenvalues of the system of equation to alleviate the numerical stiffness encountered in low-speed flows. Therefore it can accelerate the convergence to a steady state, and unify the solving algorithms at all Mach numbers. In addition modifying the numerical dissipation part of the inviscid flux according to precondition, resulting in better accuracy of the solution.

The artificial compressibility method of Chorin[1] is the original idea of low-speed precondition; subsequently many preconditioning methods were developed in the past several decades[2),3),4)]. In this paper the preconditioning method introduced by Weiss and Smith combined with full multigrid method is implemented to solve the Reynolds averaged N-S equations for viscous and inviscid flows at low Mach numbers

Low-speed Preconditioning

The Navier-Stokes equations in general coordinates have the following forms:

$$\frac{\partial}{\partial t}\int_{\Omega} \mathrm{W}\, dV + \oint_{\partial\Omega} (\mathrm{F_c} - \mathrm{F_v})\, \mathrm{d}A = 0 \qquad (1)$$

$$\mathrm{W} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, \quad \mathrm{F}_c = \begin{bmatrix} \rho U \\ \rho u U + n_x p \\ \rho v U + n_y p \\ \rho w U + n_z p \\ \rho H U \end{bmatrix}, \quad F_v = \begin{bmatrix} 0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \varphi \end{bmatrix}$$

$$U = n_x u + n_y v + n_z w,$$

$$\sigma_1 = n_x \tau_{xx} + n_y \tau_{xy} + n_z \tau_{xz}$$

$$\sigma_2 = n_x \tau_{yx} + n_y \tau_{yy} + n_z \tau_{yz}$$

$$\sigma_3 = n_x \tau_{zx} + n_y \tau_{zy} + n_z \tau_{zz}$$

$$\varphi = u\sigma_1 + v\sigma_2 + w\sigma_3 - q$$

$$q = -\kappa\left(n_x \frac{\partial T}{\partial x} + n_y \frac{\partial T}{\partial y} + n_z \frac{\partial T}{\partial z}\right)$$

$$\rho E = \rho H - p, \quad H = C_P T + (u^2 + v^2 + w^2)/2$$

Transforming the governing equations to primitive variables gives:

$$M\frac{\partial}{\partial t}\int_{\Omega} \mathrm{Q}\, dV + \oint_{\partial\Omega} (\mathrm{F_c} - \mathrm{F_v})\, \mathrm{d}A = 0 \qquad (2)$$

$$M = \frac{\partial W}{\partial Q} = \begin{bmatrix} \rho_p & 0 & 0 & 0 & \rho_T \\ \rho_p u & \rho & 0 & 0 & \rho_T u \\ \rho_p v & 0 & \rho & 0 & \rho_T v \\ \rho_p w & 0 & 0 & \rho & \rho_T w \\ \rho_p H - 1 & \rho u & \rho v & \rho w & \rho_T H + \rho C_P \end{bmatrix},$$

And replacing matrix M with Weiss-Smith[4)] preconditioning matrix,

$$P = \begin{bmatrix} \phi & 0 & 0 & 0 & \rho_T \\ \phi u & \rho & 0 & 0 & \rho_T u \\ \phi v & 0 & \rho & 0 & \rho_T v \\ \phi w & 0 & 0 & \rho & \rho_T w \\ \phi H - 1 & \rho u & \rho v & \rho w & \rho_T H + \rho C_P \end{bmatrix}$$

Thus the preconditioned N-S equations can be achieved:

$$P\frac{\partial}{\partial t}\int_{\Omega} \mathrm{Q}\, dV + \oint_{\partial\Omega} (\mathrm{F_c} - \mathrm{F_v})\, \mathrm{d}A = 0 \qquad (3)$$

Where $\phi = \dfrac{1 + (\gamma - 1)\beta}{a^2 \beta}$, a is speed of sound, β is the adjustable preconditioning parameter. The optimal value of β should be the square of local Mach number, However, in order to avoid singularities occurring in matrix P close to stagnation regions and in boundary layer where the Mach number approaches zero, the value of β must be bounded according to the solving conditions. Therefore, the final definition of β is,

$$\beta = Min(Max(\beta_{inv}, \beta_{vis}, kM_\infty^2), 1.0) \quad (4)$$

Where,

$$\beta_{inv} = \begin{cases} 2M^2/(1\text{-}2M^2), M < 0.5 \\ 1 \qquad , M \geq 0.5 \end{cases} \quad (5)$$

$$\beta_{vis} = \beta_{inv} / R_\Delta^2 \quad (6)$$

β_{inv} is inviscid bounding variable, when $M \geq 0.5$ the preconditioned equations return to the original forms. β_{vis} is used for viscous computation, and R_Δ is cell Reynolds number. kM_∞^2 is introduced to take into account inflow conditions; in general one can choose $k \sim 1.0$, for easy case $k \sim 0.2$, for hard case $k \sim 3-10$. In order to keep consistency with solving method of compressible flows and preserve the conservative characteristic, equation (3) is converted back to conservation form,

$$PM^{-1}\frac{\partial}{\partial t}\int_\Omega \mathrm{W}\, dV + \oint_{\partial\Omega}(\mathrm{F_c} - \mathrm{F_v})\, \mathrm{d}A = 0 \quad (7)$$

Equation (7) is the final form of preconditioned equation; except for PM^{-1} it is unchanged compared with equation (1). The Jacobian matrices of preconditioned governing equation $A = MP^{-1}\frac{\partial F_c}{\partial W}$, its eigenvalues are

$$\lambda(A) = (U, U, U, \frac{(\beta+1)U \pm c}{2})$$

Where $c = \sqrt{U^2(\beta-1)^2 + 4\beta a^2(n_x^2 + n_y^2 + n_z^2)}$. Setting proper value of β, all of the preconditioned eigenvalues can be of the same order of magnitude when Mach numbers less than unity, consequently the stiffness is eliminated under small-Mach-number circumstances. While the eigenvalues of equation (1) are $\lambda(\overline{A}) = (U, U, U, U \pm a)$, if $M_a << 1$ the value of $(U \pm a)/U$ is very large. Above is the principle of accelerating convergence of preconditioning. The preconditioned governing equations lost the physical significance of original time-derivative term, yet the converged steady solutions will not changed according to preconditioning.

For the purpose of time-accurate unsteady simulations, a dual time-stepping procedure is also introduced,

$$\frac{\partial}{\partial t}\int_\Omega \mathrm{W}\, dV + PM^{-1}\frac{\partial}{\partial \tau}\int_\Omega \mathrm{W}\, dV + \oint_{\partial\Omega}(\mathrm{F_c} - \mathrm{F_v})\, \mathrm{d}A = 0 \quad (8)$$

Where t is physical time, τ is pseudo time. The flow field at each physical time step goes through an inner pseudo-time loop where preconditioning is applied, maintaining the solutions accuracy at the physical time-step level. Applying first-order implicit discretization for pseudo time, second-order discretization for physical time, and then utilizing linearization the following discretized formulation achived,

$$(\frac{PM^{-1}}{\Delta\tau} + \frac{3\mathrm{I}}{2\Delta t} - \frac{\partial \mathrm{R}}{\partial \mathrm{W}})\Delta \mathrm{W}^m = R \quad (9)$$

or,

$$(\frac{I}{\Delta\tau} + \frac{3MP^{-1}}{2\Delta t} - MP^{-1}\frac{\partial \mathrm{R}}{\partial \mathrm{W}})\Delta \mathrm{W}^m = MP^{-1}R \quad (10)$$

Where,

$R = R' - \frac{3\,\mathrm{W}^m - 4\,\mathrm{W}^n + \mathrm{W}^{n-1}}{2\Delta t}$, and

$R' = -\frac{1}{\Omega}\oint_{\partial\Omega}(\mathrm{F_c} - \mathrm{F_v})\, \mathrm{d}A$

Numerical Method

In this paper cell-centered finite-volume method is used to discretize the governing equations; so in each cell the residual R' is calculated as following,

$$R' = -\frac{1}{\Omega_{I,J,K}}\sum(\mathrm{F_c} - \mathrm{F_v})_m\, \Delta A \quad (11)$$

Roe's[5] flux-difference splitting is used for spatial discretization of inviscid flux. The numerical flux at the cell face is obtained by using the following numerical flux formulas,

$$\mathrm{F}_c = \frac{1}{2}[\mathrm{F}_c(q_L) + \mathrm{F}_c(q_R) - \left|\widetilde{A}\right|(q_R - q_L)] \quad (12)$$

Here q_L and q_R denote left and right state variables, $\widetilde{A}$ is the preconditioned Jacobian matrix evaluated by using Roe-averaged after diagonalizing process. Namely $\left|\widetilde{A}\right| = |A(\tilde{q})|$, $|A| = T|\Lambda|T^{-1}$, where T and T^{-1} are right and left eigenvector matrices, and $|\Lambda|$ is a diagonal matrix consist of the absolute of eigenvalues. The original Roe scheme only has first-order accuracy, hence MUSCL extrapolation of Van Leer[6] with Van Albada limiter is used to achieve second or third order accuracy. This Roe scheme depicted above modified the numerical dissipation term according to preconditioning, resulting in more accurate solution for low Mach number flows. While the original counterpart have an amount of numerical dissipation that does not scale correctly when Mach number approaches zero. Central differencing is used for viscous terms; in the finite-volume system Green’s theorem is used for first derivatives evaluation.

A diagonal form of an implicit approximate-factorization algorithm7) is used to equation (10) for time integration, where only scalar tridiagnal inversions rather than block operator are implemented. This is a robust and rapid scheme, and it is popular in CFD community.

Boundary conditions related to characteristics have to be modified based on preconditioned system. A flux difference formulation[8] is implemented for the far field boundary conditions,

$$Q_b = 0.5(Q_\infty + Q_e) - 0.5[\mathrm{sign(A)}(Q_\infty - Q_e)] \quad (13)$$

Where $\mathrm{sign(A)} = T\ \mathrm{sign}[\lambda(\mathrm{A})]\ T^{-1}$, evaluated by using Roe-averaged variables, Q_∞ is freestream value and Q_e is extrapolated from interior mesh.

Multigrid Algorithms

The explicit and implicit time-stepping and iterative schemes in general can reduce the short-wavelength error components on a given grid efficiently. While the longer wavelength components are usually hardly damped. This results in a slow convergence to the steady solutions after some times of initial iterations. By introducing a sequence of successively coarser meshes, the longer wavelength components on the finest meshes becomes short-wavelength components on the coarser meshes and can be damped efficiently. Consequently the entire error is reduced quickly, and the convergence is significantly accelerated. This is the basic principles of multigrid algorithms.

In this paper full approximation storage (FAS) and full multigrid algorithms9) are implemented. For the preconditioned systems there are two choices for multigrid implementation, transferring the residuals based on the preconditioned system or the physical residuals to the next grid. That is to say, first to transfer residuals or to pre-multiplying the preconditioning matrix. In numerical experiment we found that transferring physical residuals based on equation (10) is more effectively for accelerating convergence.

With h and 2h denote fine and coarse meshes respectively, the basic multigrid process can be formulated in two meshes levels as follows:

(1) Transferring solutions and residuals to the coarse grid, and calculating the forcing function,

$$W_{2h}^{(0)} = I_h^{2h} W_h^{m+1}, \quad (Q_F)_{2h} = I_h^{2h} R_h^{m+1} - R_{2h}^{(0)}$$

(2) Calculating a new solution on coarse grid,

$$\left(\frac{I}{\Delta\tau} + \frac{3MP^{-1}}{2\Delta t} - MP^{-1}\frac{\partial \mathrm{R}}{\partial \mathrm{W}}\right)\Delta \mathrm{W}^m = MP^{-1}[R_{2h} + (Q_F)_{2h}]$$

(3) Interpolating from the coarse to the fine grid, Coarse grid correction:

$$\delta W_{2h} = W_{2h}^{m+1} - W_{2h}^{(0)}$$

Interpolation of correction:

$$W_h^{'} = W_h^{m+1} + I_{2h}^{h}(\delta W_{2h})$$

Numerical Results

Flows past RAE2822 airfoil

The first test case analyzed is inviscid flows past RAE2822 airfoil. The freestream condition is $\mathrm{M}_\infty = 0.01$, $\alpha = 1.89°$, The computational domain extend 25 chord lengths away from the airfoil and the 225×41 C-grid was created, as shown in figure 1. The flux difference boundary condition of equation (13) is used for far field. Figure 2A shows the pressure coefficient distributions computed with preconditioned system and the original system. The preconditioned result approach that of Puoti [11] very well, but the result without preconditioning is of poor quality. Figure 2B shows the convergence histories with and without preconditioning, and preconditioning plus there-level full multigrid method. Without preconditioning the convergence is very slow, and the residual cannot reach a low level. In contrast, with preconditioning convergence is accelerated. Furthermore, with both preconditioning and multigrid machine zero convergence is attained in much less iterations.

Other test cases, $\mathrm{M}_\infty =$ 0.1, 0.05, 0.001, behave essentially the same. At low Mach number the solutions of unmodified equations has lost significant accuracy while preconditioned solutions preserve accuracy. Also preconditioning can accelerate the convergence rate, and multigrid method is effective for preconditioned system. In addition, a case of $\mathrm{M}_\infty = 0.73$, $\alpha = 2.89°$ is calculated with preconditioning. The results agree well with that of Pulliam[10] and the shock wave is captured within three nodes. The convergence rate is also remained compared to unmodified system at compressible flows. This verifies that preconditioning does not degrade the accuracy of solutions and convergence characteristic for large Mach number flows.

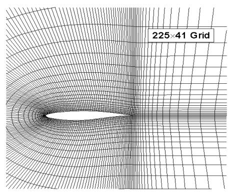
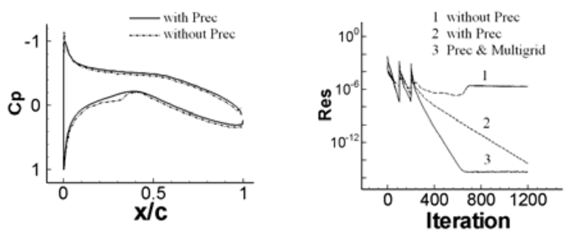


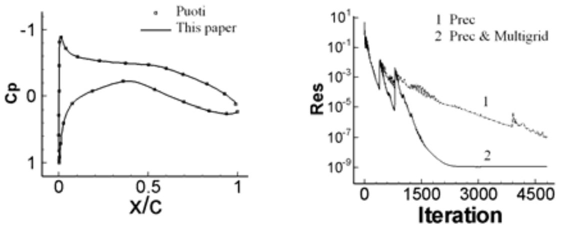
Figure 1 C-grid for the RAE2822 airfoil



A. Pressure Coefficient B. Residual

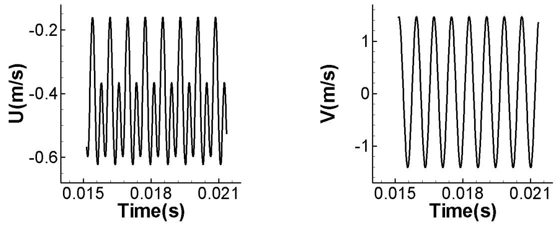
Figure 2 RAE2822 airfoil inviscid flows

For the viscous flow past RAE2822 airfoil, the conditions are $\mathrm{M}_\infty = 0.01$, $\alpha = 1.89°$, $\mathrm{Re} = 5.7\times10^6$. The 265×89 grid with clustered near the airfoil and Baldwin-Lomax turbulence model are employed in the viscous calculation. Unpreconditioned solutions are completely wrong, only preconditioned results are given. In figure 3A the comparison with results of Puoti[11] is presented, showing the accuracy of the results obtained in the low-speed viscous flows. The convergence histories with and without there-level full multigrid is presented in figure 3B. Obviously, multigrid method accelerated the preconditioning calculations effectively.



A. Pressure coefficient B. Residual

Figure 3 RAE2822 airfoil viscous flow



A. Horizontal velocity B. Vertical velocity

Figure 5 Velocity evolution

Unsteady circular cylinder flows

The viscous flow passing a circular cylinder is used for unsteady preconditioning test case. When the Reynolds number based on the freestream velocity and the cylinder diameter is larger than 40, vortex shed periodically from the circular cylinder. The computational domain extends 20 diameters around cylinder, and the 145×113 O-grid is used, as shown in figure 4. Three cases with different Reynolds number and Mach number given in Table 1 are carried out. For each period of vortex shedding, 100 physical time steps are setted. During each physical time step eight inner iterations implemented with four-level multigrid. In general, the non-dimensional frequency, namely Strouhal Number, $S_t = fD/U_0$, is used to scale the vortex shedding frequency. The computed Strouhal number compared with experimental data[12)] and the numerical results of Buelow[13)] is shown in Table 1. It is found that the numerical results are the same with the experimental data except for the case of $Re = 1000$ where the Reynolds number beyond the experimental formulation.

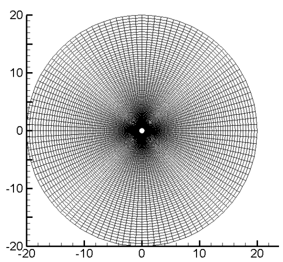


Figure 4 O-grid for the circular cylinder

Table 1 Strouhal Number

Case	Present Result	Experiment	Buelow's Result
M∞=0.02, Re=1000	0.235	0.208	0.242
M∞=0.01, Re=100	0.167	0.167	/
M∞=0.002, Re=100	0.167	0.167	/

For case of $M_\infty = 0.01$ and $Re = 100$, Figure 5 shows the time-evolution of horizontal and vertical velocity at the point one diameter away on the centerline of circular cylinder in seven periods. The instantaneous streamlines computed during one shedding period is displayed in Figure 6.

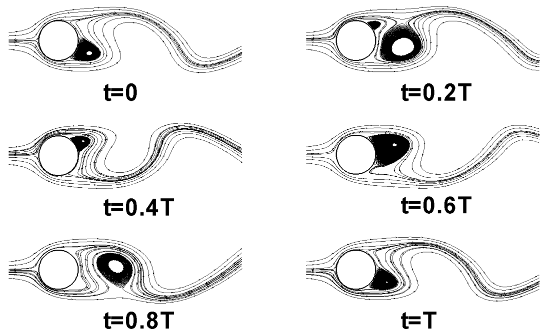


Figure 6 Instantaneous streamlines evolution within one shedding period

Conclusion

Through preconditioning technique time-marching algorithms used for compressible Navier–Stokes equations is modified to solve the low-speed flows, resulting in improvements on computational efficiency and solution accuracy. Hence the compressible solver could be used for incompressible calculation readily with minor modification. The time-accurate unsteady solution is obtained by using combined method of dual-time step and preconditioning. The multigrid algorithms introduced to preconditioning speed up the convergence further. For several inviscid-viscous, steady-unsteady test cases satisfactory results are obtained, which proven wide applicability and reliability of the algorithms implemented in this paper.

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