A HYBRID TREFFTZ FLAT SHELL ELEMENT

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Abstract
We suggest a linear elastic flat shell element based on the HT(hybrid Trefftz) method. We formulate the membrane part of the proposed element as an HT plane element with the drilling DOF. For the bending part, we developed a thick HT plate element that can represent transverse shear deformations accurately. Because we derive both the membrane and the bending parts consistently using the HT functional, we can easily construct the triangular and the quadrilateral elements in a unified way. In addition, warping of quadrilateral element is compensated by force and moment equilibrium equations. We evaluate the performance of the new element in terms of accuracy and convergence.

1. INTRODUCTION

One of the most important elements in modeling and analysis of 3-dimensional structures is a shell element. Many researchers have been interested in developing various types of high performance shell elements. Among those methods, a viable alternative is the HT method, which has little restriction on the shape of elements inherited from the variational formulation that includes only boundary integrals. Jirousek et al. developed various plane and plate elements based on the HT method. In this study, we develop an accurate and efficient HT flat shell element that can be used with conventional finite elements in the structural analysis for general purposes. We apply the HT variational principle consistently to both the membrane and the bending part of the element; the membrane part with drilling DOF's, and the bending part including transverse shear deformations based on Mindlin-Reissner's thick plate theory. Additionally, warping phenomenon of quadrilateral element is treated by force and moment equilibrium equations that consider geometrical configuration.

2. HYBRID TREFFTZ FUNCTIONAL AND ELEMENT FORMULATION

The HT functional of internal displacement u and boundary displacement $\tilde{u}$ is defined for elasticity problems as

$$J(u,\tilde{u}) = \int_{\Omega} W(u_i)d\Omega - \int_{\Omega} b_i u_i d\Omega - \int_{\Gamma_T} \overline{T}_i \tilde{u}_i d\Gamma_T - \int_{\Gamma} T_i (u_i - \tilde{u}_i) d\Gamma , \quad (1)$$

where W refers to the strain energy density function. In addition, b_i and T_i are the body force and the traction, and $\overline{T}_i$ is the prescribed traction along Γ_T. Taking variation of equation (1) with respect to the two variables and some mathematical manipulations lead to the following equations for u satisfying governing equations a priori.

$$\delta J(u,\tilde{u})\Big|_{\delta u} = -\int_{\Gamma} \delta T_i (u_i - \tilde{u}_i) d\Gamma = 0 \quad ; \quad \delta J(u,\tilde{u})\Big|_{\delta\tilde{u}} = \int_{\Gamma} T_i \delta\tilde{u}_i d\Gamma - \int_{\Gamma_T} \overline{T}_i \delta\tilde{u}_i d\Gamma = 0 \tag{2}$$

To get the finite element equations in matrix form, we should approximate u and $\tilde{u}$,

$$\{u\} = \{u_p\} + \sum_{j=1}^{m} \{\phi_j\} c_j = \{u_p\} + [\phi]\{c\} \quad ; \quad \{\tilde{u}\} = [\tilde{N}]\{d\} \tag{3}$$

where $\{u_p\}$ is a particular solution for the governing equations, and $\{\phi_j\}$, c_j, $\{d\}$ and $[\tilde{N}]$ are homogeneous Trefftz functions , their unknown coefficients, nodal DOF and 1-D shape function, respectively. Traction vector can be derived from equation (3) and constitutive equations as follows:

$$\{T\} = \{T_p\} + [\Theta]\{c\} \quad where,\ [\Theta] = [A][E][L][\phi],\ \{T_p\} = [A][E][L]\{u_p\}, \tag{4}$$

where $[L]$ is a differential operator relating displacements to strains, $[A]$ is a matrix of direction cosines of an outward normal vector, and $[E]$ represents the constitutive relation. By applying approximations in equations (3, 4) to equations (2) and eliminating the coefficient vector $\{c\}$, we get the matrix equations as follows.

$$[K]\{d\} = \{f\} \quad where \quad \begin{cases} [K] = [G]^T [H]^{-1} [G], \{f\} = \{g\} - [G]^T [H]^{-1} \{h\}, \{h\} = \int_{\Gamma} [\Theta]^T \{u_p\} d\Gamma \\ \{g\} = \int_{\Gamma} [\tilde{N}]^T \{T_p\} d\Gamma, [H] = \int_{\Gamma} [\Theta]^T [\phi] d\Gamma [G], [G] = \int_{\Gamma} [\Theta]^T [\tilde{N}] d\Gamma \end{cases} \tag{5}$$

3. APPROXIMATION OF DISPLACEMENT FIELDS FOR MEMBRANE BEHAVIOR

We construct the Trefftz function $[\phi]$ in equation (3) using the Muskehlishbili's solution for plane elasticity.

$$2\mu\Phi = 2\mu(u_x + iu_y) = \kappa F(z) - z\overline{F}'(\bar{z}) - \overline{G}(\bar{z}) \tag{6}$$

where F and G are analytic functions and over bar means complex conjugate, μ and ν are shear modulus and Poisson's ratio, and $\kappa = {}^{(3-\nu)}\!/_{(1+\nu)}$. In addition, i means an imaginary number, $z = x + iy$ and prime means differentiation with respect to z. Because Φ in equation (6) satisfy the governing equations of the plane problem, $\{u\}$ also satisfies the governing equations naturally. In the Muskehlishbili's solution, we have to select arbitrary analytic functions F and G of complex variable z. We assign complex power series to F and G as follows

$$\left.\begin{array}{l} F(z)=iz^k, F(z)=z^k, F(z)=0, F(z)=0 \\ G(z)=0, G(z)=0, G(z)=iz^k, G(z)=z^k \end{array}\right\} \quad (k=1,2,\cdots) . \tag{7}$$

However, ensuring numerical stability in forming the element stiffness matrix, we drop the first pair of F and G when $k=1$, i.e. $F(z)=iz$ and $G(z)=0$. When approximating $\{\tilde{u}\}$, we used Allman's quadratic displacement and drilling DOF. So, along an edge, the approximation can be written as

$$\begin{Bmatrix} \tilde{u}_x \\ \tilde{u}_y \end{Bmatrix} = \begin{bmatrix} \tilde{N}_{l1} & 0 & -\frac{(y_2-y_1)}{2}\tilde{N}_{l1}\tilde{N}_{l2} & \tilde{N}_{l2} & 0 & \frac{(y_2-y_1)}{2}\tilde{N}_{l1}\tilde{N}_{l2} \\ 0 & \tilde{N}_{l1} & \frac{(x_2-x_1)}{2}\tilde{N}_{l1}\tilde{N}_{l2} & 0 & \tilde{N}_{l2} & -\frac{(x_2-x_1)}{2}\tilde{N}_{l1}\tilde{N}_{l2} \end{bmatrix} \begin{Bmatrix} u_{x1} & u_{y1} & \theta_{z1} & u_{x2} & u_{y2} & \theta_{z2} \end{Bmatrix}^T , \tag{8}$$

where $\tilde{N}_{li}$ is a linear shape function on the edge, and x_i, y_i and l_{12} means the coordinates of the two end points and the length of the side, respectively. The vector $\{u_{x1} \;\; \cdots \;\; \theta_{z2}\}^T$ is a nodal DOF vector on the edge.

4. APPROXIMATION OF DISPLACEMENT FIELDS FOR BENDING BEHAVIOR

The formulation for plate element which was developed by Jiousek et al. is used for this flat shell element. The homogeneous solution for the thick plate can be constructed for the decoupled governing equations of the plate and the boundary shape functions are constructed to represent transverse shear deformations accurately. To consider the transverse shear deformations in the approximation of $\{\tilde{u}\}$, we assume that rotations are linear, while transverse displacement is quadratic by introducing a extra nodeless degree. Then we can derive the transverse shear strains from the equilibrium equations and the constitutive equations, respectively, and comparing the two strains reveals that the former is a constant while the latter is linear. Thus, setting the linear term of the latter to be zero brings the detailed expression for extra nodeless degree.

5. TREATMENT OF WARPING

Since stiffness matrix of flat shell element is calculated on transformed local coordinates that is made by three nodes, there can be warping that one of four nodes is off the transformed surface in case of quadrilateral element. For representing exact geometrical configuration, the warping phenomenon must be treated in a physically appropriate way. In this study, force and moment equilibrium are proposed to compensate the warping of quadrilateral element after calculating stiffness matrix on mid-surface of four nodes as in Fig. 1. The basic idea of this treatment is to revise couple of forces at every node on mid-surface for developing moments which exist at original configuration. At first, transverse forces f_{zi} are added to cause bending moment induced by warping in Fig. 1(a). In this case, transverse forces must satisfy self-equilibrium and moment equilibrium with external forces.

$$\sum_i f_{zi} = 0. \;\; ; \;\; \sum_i \left(f_{xi}\vec{i} + f_{yi}\vec{j} + f_{zi}\vec{k} \right) \times \vec{F}_i = 0 \tag{9}$$

Additionally, least squares method is applied to transverse forces as in Equation (10) and subsequently we can get explicit value of transverse forces.

$$Min \quad \sum_i f_{zi}^2 \qquad s.t. \quad \sum_i x_i f_{zi} = F_x; \qquad \sum_i y_i f_{zi} = F_y; \qquad \sum_i f_{zi} = 0 \tag{10}$$

Next, forces are added to cause in-plane bending moment in Fig. 1(b). These forces also satisfy moment equilibrium with original in-plane moments.

$$\sum_i \left(f_{xi}\vec{i} + f_{yi}\vec{j} \right) \times \vec{r}_i + \sum_i M_{zi} \cdot \vec{k} = 0 \tag{11}$$

As a result of revising forces at nodes, transformation matrices for forces are obtained and are applied to original stiffness matrix on mid-surface.

$$\{f_a\} = [T_M]^T \left[T_f\right]^T \{f_m\}, \quad [K_a] = [T_M]^T \left[T_f\right]^T [K_m]\left[T_f\right][T_M] \tag{12}$$

6. NUMERICAL EXAMPLES

To evaluate the performance of the proposed element (NEW3/4), we analyzed two test problems and compared the results to those of other elements. We selected TRIA3/QUAD4 of NASTRAN, Allman’s element (ALLMAN3/4) with DKMT/DKMQ element, and HT plane element (HT3/4) with HT thick plate element as comparative shell elements.

Pinched cylinder

We evaluated the performance of the shell element through the pinched cylinder problem. The reference solution is $w = 1.8248 \times 10^{-5}$ at the loading point in Fig. 2. Fig. 3 shows the normalized deflection to the reference solution. In the result, the NEW4 and NEW3 show good behavior.

Twisted beam

To evaluate the membrane and bending behavior at the same time, we solved twisted beam problem depicted in Fig. 4

with quadrilateral elements. We applied in-plane and out-of-plane unit load at the tip of the beam and listed the Normalized tip displacement with respect to the reference solution in Table 1. It is proven for the proposed elements to have good performance.

7. CONCLUSIONS

We have developed a flat shell element by combining the membrane and bending stiffness based on the HT method. We have used Allman's quadratic displacement field as a boundary displacement field to supplement drilling DOF in the membrane stiffness. We have constructed the bending stiffness from the thick plate theory and approximated the displacement field by equilibrium equations of thick plate. We have considered the transverse shear deformations using the relation of the transverse shear strains derived from the equilibrium equation and the constitutive equation. The warping phenomenon of quadrilateral element is treated mathematically considering geometrical configuration. To investigate the convergence and accuracy, we have tried a series of numerical tests. We have demonstrated accuracy and convergence rate of the proposed element.

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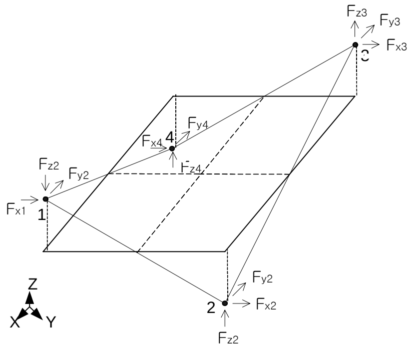
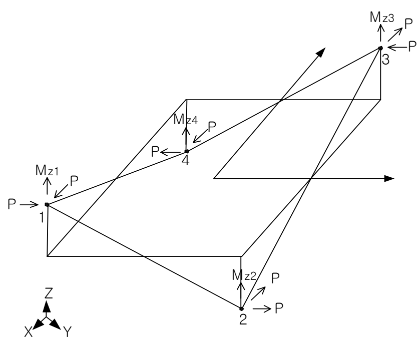


(a) Adding transverse forces



(b) Adding in-plane forces

Fig. 1 Adding couple of forces to treat warping effect

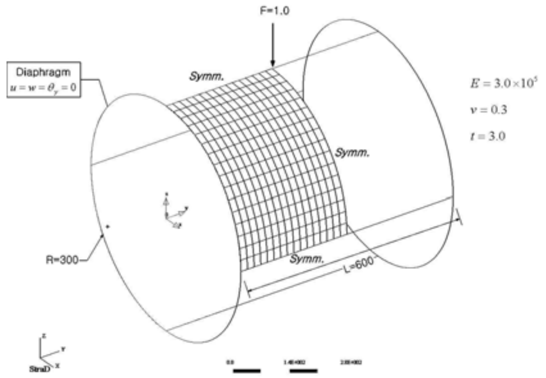
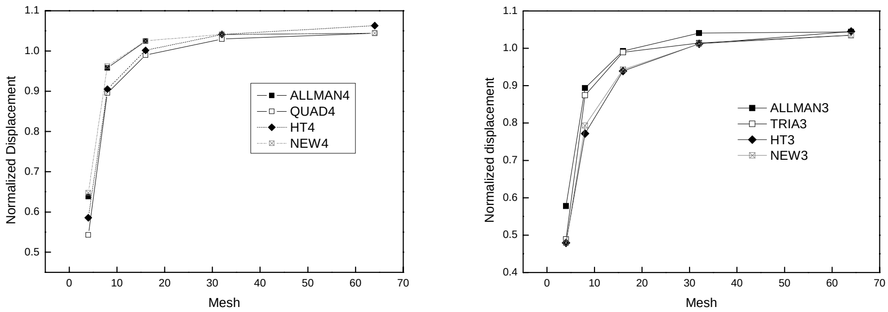


Fig. 2 Finite element model of a pinched cylinder



(a) Quadrilateral element *(b) Triangular element*

Fig. 3 Normalized displacement of the pinched cylinder

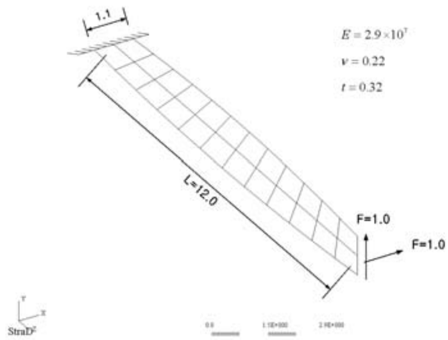


Fig. 4 Finite element model of a twisted beam

Table 1. Normalized tip displacement of a twisted beam

Mesh		ALLMAN4	QUAD4	HT4	NEW4	Ref.
in plane	1×6	0.86	0.98	5.90	0.88	0.00542
	2×12	0.95	0.99	1.35	0.97	
out of plane	1×6	1.15	0.96	24.00	1.16	0.00175
	2×12	1.06	0.99	1.49	1.11	