APPLICATION OF INTEGRODIFFERENTIAL EQUATIONS FOR THE PROBLEM OF ELECTRICALLY PERMEABLE CRACK ON A PIEZOELECTRIC-CONDUCTOR INTERFACE

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Abstract

A plane strain problem of a crack on interface between an isotropic elastic conductor and a transversely isotropic piezoelectric ceramics is considered. The problem is reduced to system integrodifferential equations on the interface. These equations relate the normal and tangential components of the crack opening vector with distribution of normal and shear stresses on the crack surfaces. It therefore make it possible to obtain an exact solution as a function of the loading applied to the crack surfaces. As an example, some analytical solutions of the crack problem are given.

INTRODUCTION

Since many electromechanical devices and instruments contain adhesion joints of piezoelectric materials, an investigation of the fracture of these joints, that induced by the interface crack propagation is of great practical importance. In particular, studying the joints of transversely isotropic piezoelectric ceramics (e.g., PZT-H5, PZT-4) with metals (e.g., steel, copper) is of special interest. The plane problem of a crack on an interface between a linear transversely isotropic piezoelectric and an isotropic elastic conductor was first considered in [1]. The axial symmetry axis of the transversely isotropic piezoelectric is normal to the interface and a normal homogeneous tensile loading is applied at infinity. In this case, electric boundary conditions specified on the crack surfaces assumed the continuity of the electric potential and the normal component of the electric displacement vector (model of an electrically permeable crack). It follows from the solution of the problem that the asymptotic behavior of the stresses in the vicinity of the crack tip is characterized by $r^{-1/2+i\beta}$, where r is the distance measured from the crack tip, β is a real number depending on the piezoelectric material constants and the conductor elastic modula, and i is the imaginary unit. As referred in [2], a general system of singular integral equations for the problem of a crack on an interface between an isotropic elastic conductor and a transversely isotropic piezoelectric was given. The solution of this system depends on the electric boundary conditions on the crack surfaces. In particular, in the case of electrically permeable boundary conditions, the problem is reduced to integrodifferential equations with Cauchy-type kernels. In the present work, typical examples using these integrodifferential equations to the problem of interface crack in the cases of simple loading conditions are shown.

FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

We consider an infinite half-space $z > 0$ of a transversely isotropic piezoelectric material in contact with an elastic isotropic conducting material occupying the half-space $z < 0$. The material constants of the piezoelectric material are $c_{11}, c_{12}, c_{13}, c_{33}, c_{44}$ (the elasticity moduli), e_{31}, e_{33}, e_{15} (the piezoelectric moduli), and $\varepsilon_{11}, \varepsilon_{33}$ (the dielectric permittivities). Young's modulus and Poisson's ratio of the conductor are given by E and ν, respectively. The contact plane $z = 0$ is the plane of isotropy for piezoelectric material. We assume that there is a crack of length $2L$ on the interface between the materials in the region $-L \le x \le L,\ -\infty < y < \infty,\ z = 0$. The case of plane strain problem has been considered. Let the composite body (piezoelectric-conductor) with an interface crack be subjected to external static mechanical and electric loadings applied outside the crack. The electroelastic state of the body (i.e., displacements, the electric field potential, the stresses, and the electric displacement vector components) can be represented by the superposition of the following states. State *A* is that of a body subjected to the same loadings in the absence of a crack. In this case, the following conditions on the interface can be specified: the conditions of mechanical contact, i.e., the continuity of the normal and tangential displacement and stresses, and the conditions of electric contact (i.e., the continuity of the electric field potential and the normal component of the electric displacement vector (in the absence of free electric charges on the interface)). State *B* is that of a body with a crack in the absence of loadings applied outside the crack. In this case, on the interface, the conditions of mechanical and electric contact are specified. On the crack surfaces, the stresses and normal component of the electric displacement vector are specified to ensure the satisfaction of the required mechanical and electric boundary conditions on the crack surfaces in the initial state. The analysis of state *A* is reduced to the solving the electroelasticity problem for a composite continuum. In this problem, the stresses and the electric displacement vector components are bounded, whereas in the vicinity of the crack tip they grow without limit. Accordingly, it is sufficient to consider state *B* for the crack problem. We assume that the loadings are applied only to the crack surfaces. We also will consider the model of an electrically permeable crack on a piezoelectric-conductor interface. This model of crack is specified by the following electric boundary conditions on the crack surfaces

$$\psi^{+} = \psi^{-}, \quad D_z^{+} = \Sigma \tag{1}$$

where D_z^{+} is the normal component of the electric displacement vector in piezoelectric medium, Σ is the charge surface density inducted in the conducting medium on the interface. The superscript + indicates the variables corresponding to the upper half-space ($z \ge 0$) and superscript $-$ indicates the variables corresponding to the lower half-space ($z \le 0$). Then, for the derivatives of normal $u_z(x)$ and shear $u_x(x)$ crack opening vector components, we have such formulations as [2]

$$\begin{aligned}\frac{d}{dx}\left[u_z(x) - i\,\gamma_1 u_x(x)\right] = \frac{1}{2g_{12}(\gamma+1)}\Bigg[&(1-\alpha)\ (\sigma_{xz}(x) + i\,\gamma_1\sigma_{zz}(x)) - \\ &-\frac{1+\alpha}{\pi\sqrt{L^2-x^2}}\left(\frac{L-|x|}{L+|x|}\right)^{-i\beta\,\mathrm{sign}(x)} \int_{-L}^{L}\frac{\sqrt{L^2-\xi^2}}{\xi-x}\left(\frac{L+\xi}{L-\xi}\right)^{-i\beta}(\gamma_1\sigma_{zz}(\xi) - i\sigma_{xz}(\xi))\,d\xi\Bigg]\end{aligned} \tag{2}$$

where $\sigma_{zz}(x)$ and $\sigma_{xz}(x)$ are normal and shear stresses on the crack surfaces, and g_{12}, γ_1, γ, α, β are real parameters, which depend on material constants of piezoelectric medium and conducting medium. For example, the values for the PZT-H5/steel joint are $g_{12} = 3.195 \times 10^9$ N/m^2, $\gamma_1 = 1.010$, $\gamma = 11.26$, $\alpha = 1.195$, $\beta = 0.028$.

The stress distribution ahead of the crack $(x > L,\ z = 0)$ in the case of an arbitrary distribution of the normal stresses $\sigma_{zz}(x)$ and shear stresses $\sigma_{xz}(x)$ on the crack surfaces has the form [2]

$$\sigma_{xz}(x) + i\gamma_1\sigma_{zz}(x) = \frac{\cosh(\pi\beta)}{\pi\sqrt{x^2 - L^2}}\left(\frac{x - L}{x + L}\right)^{-i\beta}\int_{-L}^{L}\frac{\sqrt{L^2 - \xi^2}}{\xi - x}\left(\frac{L + \xi}{L - \xi}\right)^{-i\beta}(\sigma_{xz}(\xi) + i\gamma_1\sigma_{zz}(\xi))\,d\xi \tag{3}$$

We note, that the equations (2) and (3) are very similar to the corresponding equations for a crack on the interface between two elastic materials [3,4].

For a crack of length $2L$, the expression for the stress intensity factors (SIFs) has the form [2]

$$\gamma_1 K_{\mathrm{I}} + iK_{\mathrm{II}} = \lim_{s = x - L \to +0}\sqrt{2\pi s}\,(\gamma_1\sigma_{zz}(s) + i\sigma_{xz}(s))\,s^{-i\beta} =$$
$$= -\frac{\cosh(\pi\beta)}{(2L)^{1/2+i\beta}}\sqrt{\frac{2}{\pi}}\int_{-L}^{L}\left(\frac{L + \xi}{L - \xi}\right)^{1/2+i\beta}(\gamma_1\sigma_{zz}(\xi) + i\sigma_{xz}(\xi))\,d\xi \tag{4}$$

In accordance with [5], the electroelastic energy release rate during the crack propagation is expressed by

$$G = \lim_{\Delta L \to 0}\frac{1}{2\Delta L}\int_{L}^{L+\Delta L}\left[\sigma_{xz}(x, L)\,u_x(x, L + \Delta L) + \sigma_{zz}(x, L)\,u_z(x, L + \Delta L) - D_z^{+}(x, L)\,\psi(x, L + \Delta L)\right]dx \tag{5}$$

where $\psi = \psi^{+} - \psi^{-}$ is the jump of the potential function of the electric field intensity on the crack surfaces. Using formula (1)-(4), the following equation is obtained from (5)

$$G = \frac{(\alpha + 1)\left(\gamma_1^2 K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2\right)}{8g_{12}(\gamma + 1)\gamma_1\cosh^2(\pi\beta)} \tag{6}$$

EXAMPLES OF ANALYTICAL SOLUTIONS OF THE CRACK PROBLEM

Typical examples using the equations are considered for simple loading conditions.

Example 1.

Let lumped forces separating the crack surfaces be applied at the crack center

$$\sigma_{zz}(x) = -P\delta(x),\quad \sigma_{xz}(x) = 0,\qquad |x| \le L \tag{7}$$

where $\delta(x)$ Dirac's delta function. These loading can be used for the experimental determination of the crack resistance of the adhesion joint [6]. We will construct the solution and determine energy release rate G during the crack tip propagation. Substituting (7) into (2), and integrating (2) over interval on the interval from L to x $(0 < x < L)$ we obtain the normal and shear crack opening vector components

$$u_z(x) - i\,\gamma_1 u_x(x) = \frac{(1 + \alpha)\gamma_1 L\,P}{2\pi(\gamma + 1)\,g_{12}}\int_{x}^{L}\left(\frac{L - \xi}{L + \xi}\right)^{-i\beta}\frac{d\xi}{\xi\sqrt{L^2 - \xi^2}}\ ,\qquad 0 < x \le L \tag{8}$$

Integral in left side of (8) can be expressed in terms of Appell hypergeometric function.

Substituting (7) into (3) and (4) we obtain the stress distribution ahead of the crack and SIFs

$$\sigma_{xz}(x) + i\gamma_1\sigma_{zz}(x) = i\frac{\cosh(\pi\beta)\gamma_1 \, L \, P}{\pi x\sqrt{x^2 - L^2}}\left(\frac{x-L}{x+L}\right)^{-i\beta}, \qquad x > L \tag{9}$$

$$\gamma_1 K_{\mathrm{I}} + iK_{\mathrm{II}} = \sqrt{\frac{2}{\pi}}\frac{\cosh(\pi\beta)\,\gamma_1 \, P}{(2L)^{1/2+i\beta}} \tag{10}$$

Substituting (10) into (6) we find the electroelastic energy release rate

$$G = \frac{(\alpha+1)\,\gamma_1}{8\pi g_{12}(\gamma+1)}\frac{P^2}{L} \tag{11}$$

For example, for the PZT-H5/steel joint we find $G = \left(2.25 \times 10^{-12} \ \mathrm{m^2/N}\right) P^2 / L$.

Example 2.

Assume that uniformly distributed normal stresses σ_0 are imposed on the far boundary, while constant normal and shear adhesive stresses between the edges of the crack surfaces σ_* and τ_* act in the end regions of length d adjacent to the crack tips $(L - d \le |x| \le L, \ z = 0)$. These adhesive stresses correspond to the plastic flow of the adhesive in a thin intermediate layer and satisfy a certain plasticity criterion $f(\sigma_*, \tau_*) = 0$, where f is a monotonically increasing function of the absolute values of σ_* and τ_* which depends on the adhesive properties. As mentioned before, it is sufficient to consider state *B* for the crack problem. Then, we shall consider the problem of a crack of length $2L$ with the following normal and shear stresses on the crack surfaces

$$\sigma_{zz}(x) = \begin{cases} -\sigma_0 + \sigma_*, & L-d \le |x| \le L \\ -\sigma_0, & |x| < L-d \end{cases}, \qquad \sigma_{xz}(x) = \begin{cases} \tau_*, & L-d \le x \le L \\ 0, & |x| < L-d \\ -\tau_*, & -L \le x \le -(L-d) \end{cases} \tag{12}$$

Substituting (12) into (2)-(4), the crack opening vector components, the stress distribution ahead of the crack and SIFs, analogously to what was done in [7] for the crack on the interface between two elastic materials, can be determined. The analytical results for the crack opening vector components, the stress distribution ahead of the crack and SIFs may be expressed in terms of hypergeometric ${}_2F_1$ function. Here, we will only consider energetic characteristics of the crack: energy release rate and adhesion fracture energy in the limit equilibrium state. We assume that the size of the end region is small compared with the length of the crack. Substituting (12) into (4), we have the leading term of the asymptotic expansion when $d/L << 1$

$$\begin{aligned} &\gamma_1 K_{\mathrm{I}} + iK_{\mathrm{II}} = \gamma_1 K_{\mathrm{I0}} + iK_{\mathrm{II0}} - (\gamma_1 K_{\mathrm{I*}} + iK_{\mathrm{II*}}), \\ &\gamma_1 K_{\mathrm{I0}} + iK_{\mathrm{II0}} = \frac{1+2i\beta}{2^{i\beta}}\sqrt{\pi}\,\gamma_1\sigma_0 \, L^{1/2-i\beta}, \\ &\gamma_1 K_{\mathrm{I*}} + iK_{\mathrm{II*}} = \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{(1+2i\beta)\cosh(\pi\beta)}{1+4\beta^2}(d/L)^{1/2-i\beta} L^{1/2-i\beta}(\gamma_1\sigma_* + i\tau_*) \end{aligned} \tag{13}$$

where $K_{\mathrm{I}0}$, $K_{\mathrm{II}0}$ are SIFs due to the action of the external load σ_0 , $K_{\mathrm{I}*}$, $K_{\mathrm{II}*}$ are SIFs due to the adhesive stresses acting in the end region of the crack.

We will now consider the limit equilibrium state of crack, which is characterized by the action of the adhesive forces in the end region of the crack and for which there is no energy flux through the points $x = \pm L$ accompanying the crack growth. The case in which total SIFs K_{I}, K_{II} are equal to zero corresponds to this example. Equating expression (13) to zero, we obtain a relation which connects the external load, the adhesive stresses, length of crack and the size of the end region of the crack in the limit equilibrium state.

$$\gamma_1\sigma_* + i\tau_* = \frac{\pi(1+4\beta^2)\ \gamma_1}{2\sqrt{2}\cosh(\pi\beta)\sqrt{d/L}}\left(\frac{d}{2L}\right)^{i\beta}\sigma_0 \tag{14}$$

If, in accordance with expression (14), σ_*, τ_* is substituted into the plastic flow criterion $f(\sigma_*, \tau_*) = 0$, it is possible to construct the function *d* from σ_0/σ_{*Y} (the value of σ_{*Y}, which is determined from the plastic flow criterion $f(\sigma_{*Y}, 0) = 0$, is the yield point of the adhesive) and to compare it with the experimental curve, which can be constructed using measurements of the size of plastic zone *d* and the limit load σ_0 as was done in [8] in the case of cracks in homogeneous materials.

The equality

$$G_0 = G_* \tag{15}$$

holds in the limit equilibrium state, where

$$G_0 = \frac{(\alpha+1)\left(\gamma_1{}^2 K_{\mathrm{I}0}{}^2 + K_{\mathrm{II}0}{}^2\right)}{8g_{12}(\gamma+1)\gamma_1\cosh^2(\pi\beta)} = \frac{\pi(\alpha+1)(1+4\beta^2)\gamma_1}{8g_{12}(\gamma+1)\cosh^2(\pi\beta)}\sigma_0{}^2 L \tag{16}$$

is the electroelastic energy release rate due to the action of the external load σ_0 and

$$G_* = \frac{(\alpha+1)\left(\gamma_1{}^2 K_{\mathrm{I}*}{}^2 + K_{\mathrm{II}*}{}^2\right)}{8g_{12}(\gamma+1)\gamma_1\cosh^2(\pi\beta)} = \frac{(\alpha+1)(\gamma_1{}^2\sigma_*{}^2 + \tau_*{}^2)\ L\ d}{\pi g_{12}(\gamma+1)\gamma_1(1+4\beta^2)} \tag{17}$$

is the critical value of the rate of energy absorption at the crack tip (the adhesion fracture energy).

The condition of limit stretching in the edge of the end region

$$\left|u_z(L-d) - iu_x(L-d)\right| = \delta \tag{18}$$

where δ is a constant which characterizes the material of the adhesive layer, is one of the conditions used to estimate the size of the end region *d*. Substituting (12) and (14) into (2), we have the leading term of the asymptotic expansion when $d / L << 1$ for the components of the vector for the crack opening at the edge of the end region $x = L - d$

$$u_z(L-d) - i\gamma_1 u_x(L-d) = \frac{(\alpha+1)}{\pi g_{12}(\gamma+1)} \frac{1+2i\beta}{1+4\beta^2} (\gamma_1\sigma_* - i\tau_*)\, d \tag{19}$$

Then taking into account, that for real piezoceramic-metal joints $\beta \approx 0,\ \gamma_1 \approx 1$, we obtain from (17)-(19) the relation

$$G_* \approx \sqrt{{\sigma_*}^2 + {\tau_*}^2}\ \delta \tag{20}$$

which associates the magnitude of the rate of energy absorption (the adhesion fracture energy) with the stresses acting in the end regions and with the limit stretching.

CONCLUSIONS

In the present work, the integrodifferential equations for the plane strain problem of an electrically permeable crack on interface between an isotropic elastic conductor and a transversely isotropic piezoelectric ceramics were considered. These equations allow us to obtain an exact solution of the crack problem as a function of the loading applied to the crack surfaces. Examples of analytical solutions of the crack problem for simple loading conditions were given. In particular, such solutions can be used for calculation of the energy release rate and prediction of the adhesion fracture energy of the piezoceramic/metal joints.

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