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EXCEL Tools for Geotechnical Reliability Analysis

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SYNOPSIS: This paper discusses two user-friendly reliability techniques that could be implemented easily using the ubiquitous EXCEL. The techniques are First-Order Reliability Method with non-Gaussian random variables expressed using Hermite polynomials and collocation-based stochastic response surface method. It is believed that ease of implementation would popularize use of reliability-based design in practice.

Key words: Reliability analysis, First-order reliability method, Hermite polynomials, Collocation-based stochastic response surface method, EXCEL computational tools.

1. Introduction

It is well accepted that uncertainties in geotechnical engineering design are unavoidable and numerous practical advantages are realizable if uncertainties and associated risks can be quantified. This is recognised in a recent National Research Council (2006) report on *Geological and Geotechnical Engineering in the New Millennium: Opportunities for Research and Technological Innovation*. The report remarked that "paradigms for dealing with … uncertainty are poorly understood and even more poorly practiced" and advocated a need for "improved methods for assessing the potential impacts of these uncertainties on engineering decisions …." Within the arena of design code development, increasing regulatory pressure is compelling geotechnical Load and Resistance Factor Design (LRFD) to advance beyond empirical re-distribution of the original global factor of safety to a simplified reliability-based design (RBD) framework that is compatible with structural design. RBD calls for a willingness to accept the fundamental philosophy that: (a) absolute reliability is an unattainable goal in the presence of uncertainty and (b) probability theory can provide a formal framework for developing design criteria that would ensure that the probability of "failure" (used herein to refer to exceeding of any prescribed limit state) is acceptably small. Ideally, geotechnical LRFD should be derived as the logical end-product of a philosophical shift in mindset to probabilistic design in the first instance and a simplification of rigorous reliability-based design into a familiar "look and feel" design format in the second. The need to draw a clear distinction between accepting reliability analysis as a necessary theoretical basis for geotechnical design and downstream calibration of simplified multiple-factor design formats, with emphasis on the former, was highlighted by

Phoon et al. (2003). The former provides a consistent method for propagation of uncertainties and a unifying framework for risk assessment across disciplines (structural and geotechnical design) and national boundaries. Other competing frameworks have been suggested (e.g., Taylor series method by Duncan 2000) but none has the theoretical breadth and power to handle complex real-world problems that may require nonlinear 3-D finite element or other numerical approaches for solution.

In conjunction with regulatory pressure to harmonize structural and geotechnical design codes, the push towards greater economic cooperation and integration through multilateral and/or bilateral free trade agreements will require the elimination of some technical obstacles that exist as a consequence of differences in national codes and standards. One on-going example of this code harmonisation phenomenon is the development of the Eurocodes within the European Union. The Eurocodes were initiated by the Commission of the European Communities (CEC) as a development of the Construction Products Directive that requires a series of harmonised European Standards to provide certain "Essential Requirements" of safety, economy, and fitness for use but which do not hinder trade within the European Community (Simpson and Driscoll, 1998). It is quite clear that the focus of harmonisation is to facilitate trade and perhaps more importantly, to ensure fair competition in the construction industry between member nations. Similar harmonisation efforts currently are underway in Canada (Green and Becker, 2001) and Japan (Honjo and Kusakabe, 2002).

The system of Structural Eurocodes consists of 10 standards. The basis for design is laid out in the head Eurocode EN1990:2002 that describes the principles and requirements for safety, serviceability and durability of structures, the basis for their design and verification, and gives guidelines for related aspects of structural reliability. It suffices to note here that reliability analysis is the only available theoretical tool capable of ensuring consistent safety between different materials. Although Eurocode 7 (Geotechnical design) adheres to the limit state design concept, it is not as well integrated as the other material codes because its partial factors are essentially empirical and precedence-based, rather than reliability-based. The important point here is that code harmonization also entails harmonization between structural and geotechnical design. Because geotechnical design is only one component of the Structural Eurocodes, it is anticipated that structural reliability methods will eventually prevail in Eurocode 7.

There is an urgent need for geotechnical engineers to play a more active role in the development of reliability-based design (RBD) methodology. Currently, most of the impetus in the development of RBD codes arises from within the structural engineering community. However, research in geotechnical reliability that addresses issues relevant to the geo-profession is progressing quite rapidly in recent years. Several specialty workshops have been organized in recent years. They are the International Workshop on Foundation Design Codes and Soil Investigation in view of International Harmonization and Performance Based Design (Honjo et al., 2002), International Workshop on Limit State design in Geotechnical Engineering Practice (Phoon et al., 2003), International Workshop on Risk Assessment in Site Characterization and Geotechnical Design (Sivakumar Babu and Phoon, 2004), International Workshop on Evaluation of Eurocode 7 (Orr, 2005), International Symposium on New Generation Design Codes for Geotechnical Engineering Practice (Lin et al., 2006), and First International Symposium on Geotechnical

Safety and Risk (Huang and Zhang, 2007). A more complete list of past activities from 1971 is given at www.geoengineer.org/reliability.

Despite emerging indications that regulatory pressure will eventually bring geotechnical design within a reliability framework and the growing interest in the research community, it is accurate to say that the average practitioner is largely unfamiliar with RBD and its potential benefits. One reason is that most research papers do not provide computational steps with sufficient details for the lay person to implement reliability analysis using common desktop softwares. In short, there is a lack of user-friendly tools. Low and Tang (2004) demonstrated that simple geotechnical reliability problems could be analyzed easily using EXCEL, but their emphasis on pedagogy is rare within a research culture that values originality. Phoon (2008a) advocated that "*we are now at the point where RBD really can be used as a rational and practical design mode. The main impediment is not theoretical (lack of power to deal with complex problems) or practical (speed of computations, availability of soil statistics, etc.), but the absence of simple computational approaches that can be easily implemented by practitioners. Much of the controversies reported in the literature are based on qualitative arguments. If practitioners were able to implement RBD easily on their PCs and calculate actual numbers using actual examples, they would gain a concrete appreciation of the merits and limitations of RBD. Misconceptions will be dismissed definitively, rather than propagated in the literature, generating further confusion. The author believes that the introduction of powerful but simple-to-implement approaches will bring about a greater acceptance of RBD amongst practitioners in the broader geotechnical engineering community*."

The objective of this paper is to briefly outline two user-friendly reliability techniques that could be readily implemented in EXCEL. "User-friendly" methods refer to those that can be implemented on a modest desktop PC by a non-specialist with limited programming skills; in other words, methods within reach of the general practitioner. Section 2 discusses how non-Gaussian random variables can be represented using Hermite polynomials in a completely general way, even if the data do not fit any known classical probability distributions. This generality is useful for practitioners with limited knowledge of available probability distributions. The First-Order Reliability Method (FORM) is computationally simpler and more robust when used in conjunction with Hermite polynomials than the commonly used equivalent Gaussian technique. Extension to correlated random vectors is quite straightforward but covered elsewhere (Phoon and Nadim, 2004). Section 3 discusses how small probabilities of failure can be computed efficiently using the collocation-based stochastic response surface method (CSRSM) method, which is more powerful and general than the popular First-Order Reliability Method (FORM). A more comprehensive presentation of RBD methods and applications targeted to the general practitioner is given in a recently published book "*Reliability-Based Design in Geotechnical Engineering: Computations and Applications*" by Taylor & Francis. Chapter 1 provides a primer on numerical recipes for reliability analysis (Phoon, 2008a). Other computational methods are presented in Chapter 3 (Low, 2008), Chapter 4 (Honjo, 2008) and Chapter 7 (Sudret and Berveiller, 2008). The spatial variability of geomaterials is one of the distinctive aspects of geotechnical RBD. This important aspect is covered in Chapter 2 (Baecher and Christian, 2008). Geotechnical examples illustrating reliability analyses and design are provided in the rest of the eight chapters. By explaining RBD with emphasis on "how to calculate" and "how to apply", this book would help to reduce the misconception that reliability analysis is "too complicated" for the general practitioner and "too impractical" for routine designs.

2. First-order Reliability Method Using Hermite Polynomials

Structural reliability theory has a significant impact on the development of modern design codes. Much of its success could be attributed to the advent of the first-order reliability method (FORM) – which provides a practical scheme of computing small probabilities of failure at high dimensional space spanned by the random variables in the problem. The basic theoretical result was given by Hasofer and Lind (1974). With reference to time-invariant reliability calculation, Rackwitz (2001) observed that: "For 90% of all applications this simple first-order theory fulfills all practical needs. Its numerical accuracy is usually more than sufficient".

The general reliability problem consists of a performance function $P(x_1, x_2, \ldots, x_n)$ and a multivariate probability density function $f(x_1, x_2, \ldots, x_n)$. The former is defined to be zero at the limit state, less than zero when the limit state is exceeded ("fail"), and larger than zero otherwise ("safe"). The performance function is nonlinear for most practical problems. The latter specifies the likelihood of realizing any one particular set of input parameters $(x_1, x_2, \ldots, x_n)$, which could include material, load, and geometrical parameters. The objective of reliability analysis is to calculate the probability of failure, which can be expressed formally as follow:

$$p_f = \int_{P<0} f(x_1, x_2, \cdots, x_n)\, dx_1 dx_2 \cdots dx_n \qquad (1)$$

The domain of integration is illustrated by a shaded region in the left panel of Fig. 1a. Exact solutions are not available even if the multivariate probability density function is normal unless the performance function is linear or quadratic. In principle, Monte Carlo simulation can be used to solve Eq. (1), regardless of the complexities underlying the physical model and/or input uncertainties. It is certainly justifiable to say that solution of a multi-dimensional integral is too complicated. Monte Carlo simulation is simple but notoriously tedious to the verge of impractical for large soil-structure interaction problems requiring finite element solutions.

Fortunately, FORM is simple and practical. The approximate solution obtained from FORM is easier to visualize in a standard space spanned by uncorrelated Gaussian random variables with zero mean and unit standard deviation (Fig. 1b). If one replaces the actual limit state function (P = 0) by an approximate linear limit state function (P_L = 0) that passes through the most likely failure point (also called design point or β-point), it follows immediately from rotational symmetry of the circular contours that:

$$p_f \approx \Phi(-\beta) \qquad (2)$$

The practical result of interest here is that Eq. (1) simply reduces to a constrained nonlinear optimisation problem:

$$\beta = \min \sqrt{\underline{u}'\underline{u}} \qquad \text{for } \{\underline{u}: P(\underline{u}) \leq 0\} \qquad (3)$$

in which $\underline{u} = (u_1, u_2, \ldots, u_n)^T$. The solution of a constrained optimization problem is significantly cheaper than the solution of a multi-dimensional integral [Eq. (1)]. It is often cited that the β-point is the "best" linearization point because the probability density is highest at that point. In actuality, the choice of the β-point requires asymptotic analysis (Breitung, 1984). In short, FORM works well only for sufficiently large β - the usual rule-of-thumb is $\beta > 1$ (Rackwitz, 2001).

Low and co-workers (e.g., Low, 2008) demonstrated that the SOLVER function in EXCEL can be easily implemented to calculate the first-order reliability index for a range of practical problems. The key advantages to applying SOLVER for the solution of Eq. (3) are: (a) EXCEL is available in almost all PCs, (b) most practitioners are familiar with the EXCEL user interface, and (c) no programming skills are needed if the performance function can be calculated using EXCEL built-in mathematical functions.

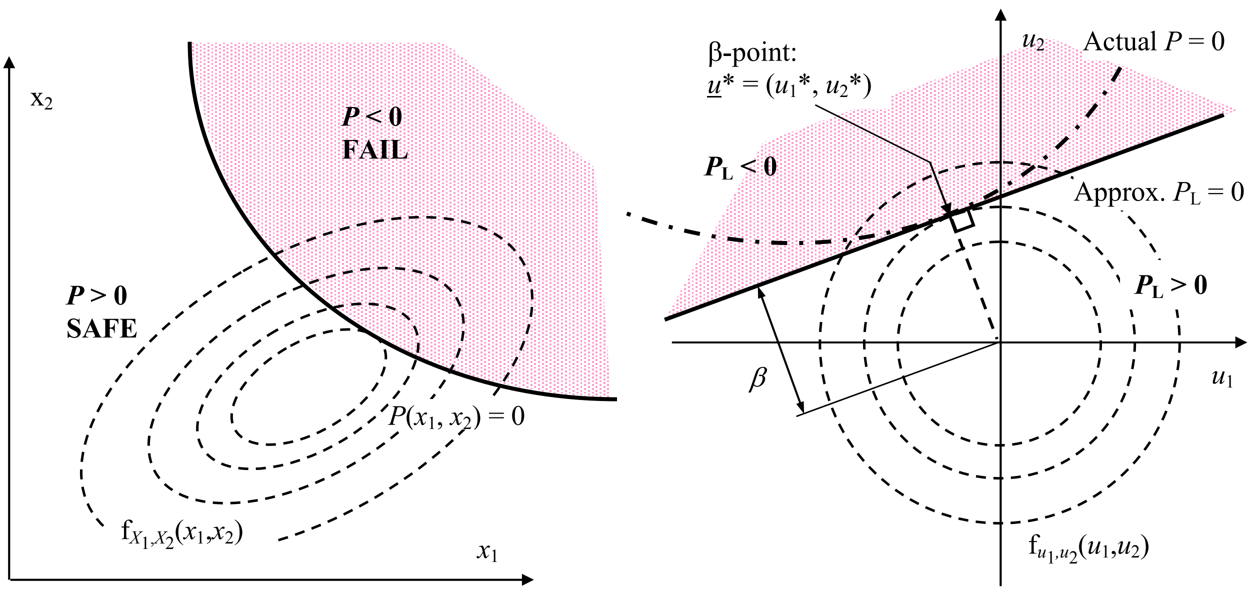


Fig. 1 (a) General reliability problem and (b) solution using FORM.

2.1 Hermite Polynomials

Hermite polynomials are given by:

$$\begin{aligned} H_0(U) &= 1 \\ H_1(U) &= U \\ H_2(U) &= U^2 - 1 \\ H_3(U) &= U^3 - 3U \\ H_{k+1}(U) &= U\,H_k(U) - k\,H_{k-1}(U) \end{aligned} \qquad (4)$$

where U is a standard Gaussian random variable. The last row of Eq. (4) shows that Hermite polynomials of any degree (k) can be computed efficiently using a simple recurrence relation that depends only on two preceding Hermite polynomials. This recurrence relation can be implemented directly using EXCEL. It can be proven rigorously (Phoon, 2003) that any random variable X (with finite variance) can be expanded as follows:

$$X = \sum_{k=0}^{\infty} a_k H_k(U) \qquad (5)$$

The key practical advantage of Eq. (5) is that the randomness of X is completely accounted for by the randomness of U, which is a known random variable.

The numerical values of the coefficients, a_k, depend on the distribution of X. In principle, the coefficients are obtained from the following integral (Phoon, 2003):

$$a_k = \frac{1}{k!}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} F^{-1}\Phi(u) H_k(u)\,du \qquad (6)$$

where F^{-1} is the inverse cumulative distribution function of X and Φ is the cumulative distribution function of U. This is another example where a theoretically elegant solution needs to be approximated to make the calculation steps more "user-friendly". Fortunately, in this case, the coefficients can be computed in a simpler way using EXCEL (Berveiller et al., 2004):

1. Let $\underline{x}$ be a $n \times 1$ vector containing measured data or simulated data from a known cumulative distribution function, F(x).
2. Let $\underline{u} = \Phi^{-1}F(\underline{x})$ be a $n \times 1$ vector containing n realizations of a standard Gaussian random variable. The function Φ^{-1} can be invoked using NORMSINV in EXCEL.
3. Let $\underline{h}_0$ be a $n \times 1$ vector containing ones, $\underline{h}_1 = \underline{u}$, $\underline{h}_2 = \underline{u} \cdot\!* \underline{u} - 1$, … $\underline{h}_{p\text{-}1} = \underline{u}\cdot\!*\underline{h}_{p\text{-}2} - (p\text{-}2)\underline{h}_{p\text{-}3}$, and H be a $n \times p$ matrix containing $\underline{h}_0$, $\underline{h}_1$, $\underline{h}_2$, … $\underline{h}_{p\text{-}1}$ in the columns. The operator "·*" means element-wise matrix

multiplication (MATLAB convention), i.e., for matrix $A = B\cdot{*}C$, (i, j) element in A, $a_{ij} = b_{ij} \times c_{ij}$. The practical nicety here is that the above recurrence relation and element-wise multiplication are automatically implemented when you "copy" & "paste" formula in EXCEL.

4. Let $\underline{a}$ be a $p \times 1$ vector containing the unknown Hermite coefficients $\{a_0, a_1, a_2, \ldots a_{p-1}\}^T$. This vector is computed by solving the following system of linear equations:

$$(H^T H)\underline{a} = H^T \underline{x} \qquad (7)$$

Eq. (7) can be solved easily using array formulae and matrix functions in EXCEL (TRANSPOSE, MMULT, MINVERSE).

The above calculation steps can be used, even if the empirical distribution of X cannot be fitted to any classical probability distribution functions. It is worth re-iterating that "Hermite polynomials" sounds more intimidating than the actual computation. The polynomials are computed by simply recurring an appropriate formula in EXCEL. The coefficients are evaluated using standard methods for solving simultaneous linear equations. The spreadsheet is given in Fig. 9-5, Phoon (2008b).

2.2 Convergence in Probability Tails

The accuracy of Eq. (5) can be studied using the lognormal distribution. By definition, X is lognormally distributed if $\ln(X)$ is normally distributed with mean = λ and variance = ξ^2. The Hermite coefficients are available in closed-form (Berveiller et al. 2004):

$$a_k = \frac{\xi^k}{k!}\exp\left(\lambda + \frac{\xi^2}{2}\right) \qquad (8)$$

It is relatively easy to proof Eq. (8) using Eq. (6). Fig. 2 shows the magnified tail probabilities of two lognormal distributions using a vertical log-scale. An eight-term Hermite expansion is sufficient to match probabilities as low as 10^{-5}, even for a strongly skewed lognormal distribution with $\lambda = 0$ and $\xi = 1.0$.

Structural reliability problems generally involve random variables with coefficients of variation less than 0.3 ($\xi < 0.3$ for lognormal case). When variabilities are much larger (Phoon and Kulhawy, 1999), the modified four-term Hermite expansion found in structural reliability software such as STRUREL (1991) may not be sufficient. In particular, significant errors can appear in the distribution tail, which is more relevant in reliability analysis. Table 1 shows the simulated moments of random variables defined by Hermite expansions. As to be expected, all moments with the exception of the mean are functions of expansion length.

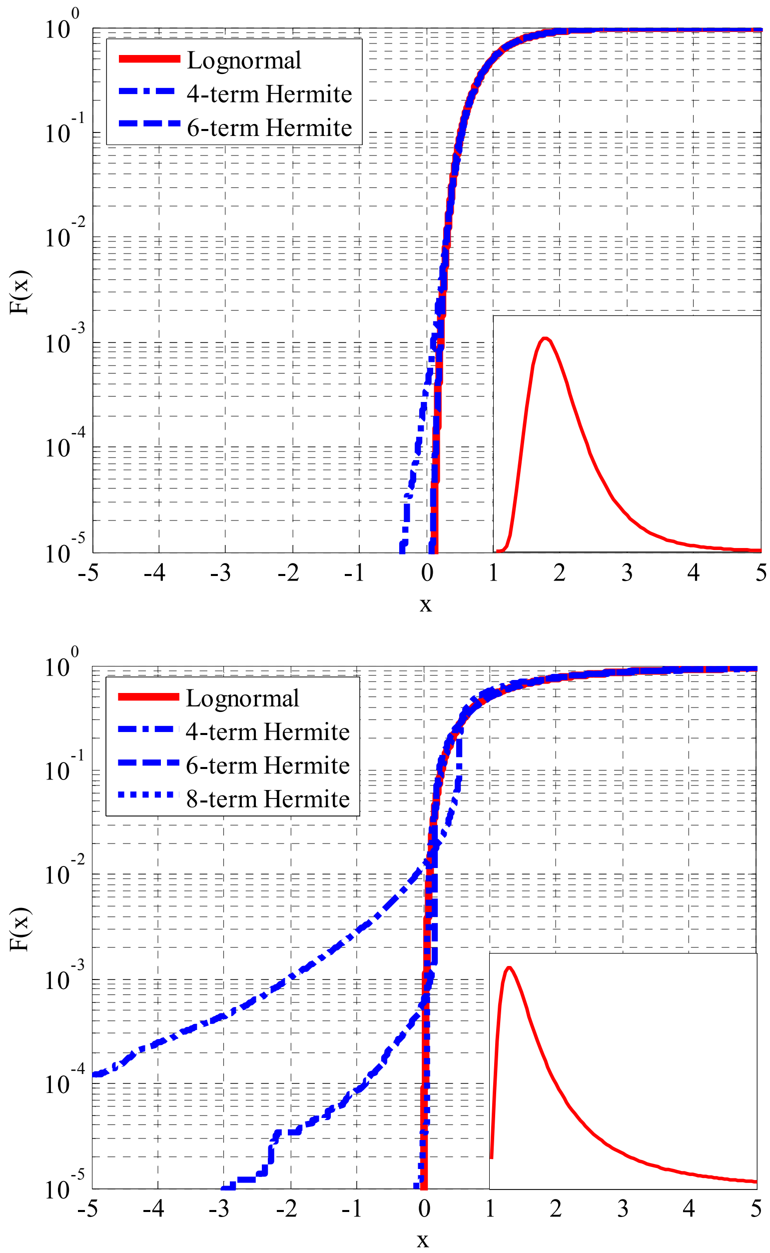


Fig. 2. Probability tails of Hermite expansions for lognormal distribution with:
$\lambda = 0$ and $\xi = 0.5$ (top); $\xi = 1.0$ (bottom).

Table 1. Moments of Hermite expansions based on simulation.

Random variable	Statistics	ξ		
		0.25	**0.5**	**1**
Lognormal (Analytical solutions)	Mean	1.03	1.13	1.65
	Variance	0.069	0.36	4.67
	Skewness	0.78	1.75	6.18
	Kurtosis	4.09	8.90	113.94
(Simulation based on sample size = 500000)				
Lognormal	Mean	1.03	1.13	1.65
	Variance	0.068	0.36	4.62
	Skewness	0.77	1.74	6.32
	Kurtosis	4.08	8.87	121.42
2-term Hermite	Mean	1.03	1.13	1.65
	Variance	0.066	0.32	2.71
	Skewness	0.00	0.00	0.00
	Kurtosis	3.00	3.00	3.00
4-term Hermite	Mean	1.03	1.13	1.65
	Variance	0.069	0.36	4.49
	Skewness	0.77	1.68	3.94
	Kurtosis	4.05	7.92	29.19
6-term Hermite	Mean	1.03	1.13	1.65
	Variance	0.068	0.36	4.61
	Skewness	0.77	1.74	5.89
	Kurtosis	4.08	8.84	87.95
8-term Hermite	Mean	1.03	1.13	1.65
	Variance	0.068	0.36	4.62
	Skewness	0.77	1.74	6.29
	Kurtosis	4.08	8.87	117.88

2.3 Application to First-Order Reliability Method

This section demonstrates that FORM used in conjunction with Hermite polynomials is computationally simpler and more robust than the commonly used equivalent Gaussian technique. The reason is that FORM iterations can take place fully in standard Gaussian space, where all variables are scaled to unit variance. The example considered is a footing subjected to a vertical dead load and a horizontal live load in sand as shown in Fig. 3. The performance function (P) and the input parameters are given in Tables 2 and 3, respectively. The mean factor of safety calculated based on the mean input parameters is 2.87.

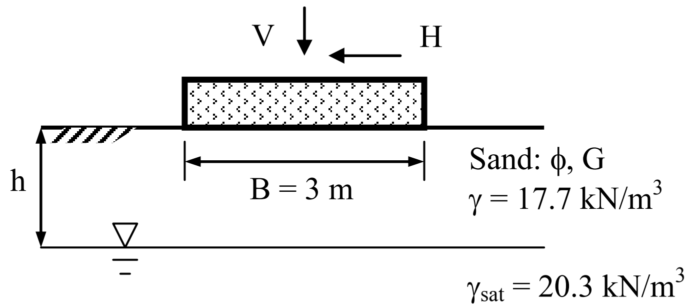


Fig. 3. Footing problem subjected to inclined loading.

Table 2. Performance function, P, for footing problem.

h = depth of groundwater table below ground surface
γ and γ_{sat} = moist unit weight and saturated unit weight of sand, respectively
γ_w = unit weight of water (9.81 kN/m^3)
ϕ = effective stress friction angle of sand
G = shear modulus of sand
V = vertical dead load
H = horizontal live load

$P = (0.5\, B\, \gamma^*\, N_\gamma\, \zeta_{\gamma s}\, \zeta_{\gamma i}\, \zeta_{\gamma r})\, B^2 - V$

$N_q = \exp(\pi \tan \phi) \tan^2(45^o + \phi/2)$
$N_\gamma = 2\,(N_q + 1) \tan \phi$

$\gamma' = \gamma_{sat} - \gamma_w = 20.3 - 9.81 = 10.5$ kN/m^3

γ^* (kN/m^3)	$= \gamma = 17.7$	$h > B$
	$= \gamma' + (\gamma - \gamma')h/B = 10.5 + 7.2h/B$	$B > h > 0$

$\zeta_{\gamma s} = 1 - 0.4\,(B/L) = 0.6;\quad \zeta_{\gamma i} = \left(1 - \frac{H}{V}\right)^{2.5}$

Rigidity index, $I_r = G / (\sigma'_a \tan \phi)$

Reduced rigidity index, $I_{rr} = I_r / (1 + I_r\, \Delta)$

Δ	$= 0.00025\,(45 - \phi)(\sigma'_a / 100\text{ kPa})$	(Note: ϕ in degrees)
σ'_a (kPa)	$= 0.5B\gamma = 26.55$	$h > B/2$
	$= h\gamma + (0.5B - h)\gamma' = 7.2h + 5.25B$	$B/2 > h > 0$

$I_{rc} = 0.5 \exp[(3.30 - 0.45\, B/L) \cot(45^o - \phi/2)]$

$I_{rr} > I_{rc} \Rightarrow$ General shear failure
$I_{rr} < I_{rc} \Rightarrow$ Local/punching shear failure

$\zeta_{\gamma r} = \exp\{[(-4.4 + 0.6\, B/L) \tan \phi] + [(3.07 \sin \phi)(\log_{10} 2I_{rr})/(1 + \sin \phi)]\}$	$I_{rr} < I_{rc}$
$= 1$	otherwise

Table 3. Description of input parameters.

Variable	Description	Distribution	Statistics
h	Depth of water table	Lognormal	mean = 2 m cov = 50%
ϕ	Effective stress friction angle	Lognormal	mean = 35^{o} cov = 8%
G	Shear modulus	Lognormal	mean = 20 MPa cov = 50%
V	Vertical dead load	Normal	mean = 1500 kN cov = 5%
H	Horizontal live load	Extreme Type I	mean = 150 kN cov = 20%
B	Footing width	Deterministic	3 m
γ	Moist unit weight of soil	Deterministic	17.7 kN/m^3
γ_{sat}	Saturated unit weight of soil	Deterministic	20.3 kN/m^3
γ_w	Unit weight of water	Deterministic	9.81 kN/m^3

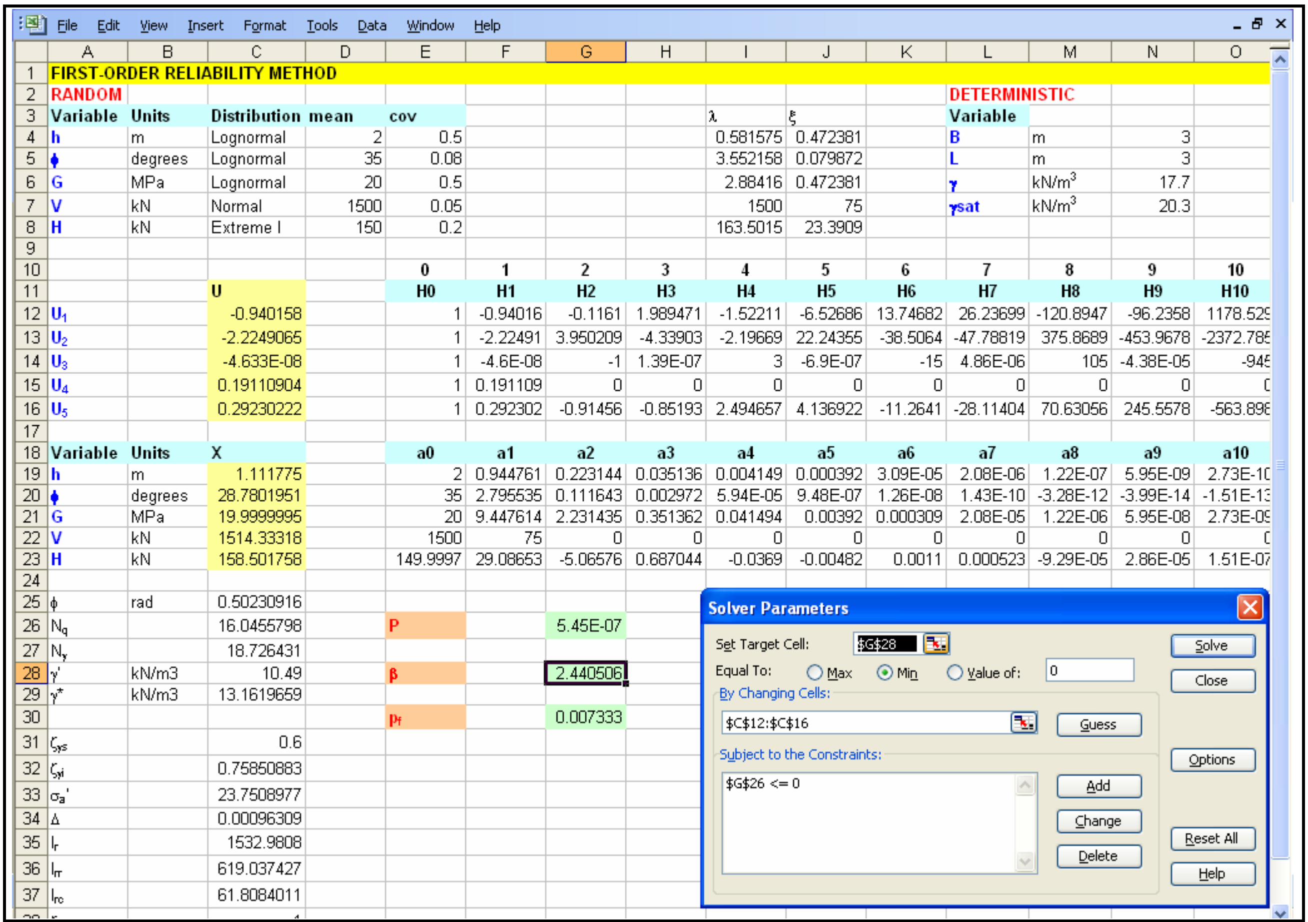
$\gamma = \gamma_w\ (G_s + 0.2e)/(1+e)$ (assume degree of saturation = 20% for "moist") = 17.7 kN/m^3
$\gamma_{sat} = \gamma_w\ (G_s + e)/(1+e)$ (degree of saturation = 100%) = 20.3 kN/m^3
Assume specific gravity of solids = G_s = 2.6 and void ratio = e = 0.5

As shown in Fig. 1, it is important to convert the non-Gaussian physical variables into standard Gaussian random variables. The conventional method is to use the equivalent Gaussian technique, which involves converting the physical variate (x_i) into the standard Gaussian variate (u_i) as follows:

$$u_i = \frac{x_i - \mu_i^N}{\sigma_i^N} = \frac{x_i - \{x_i - \sigma_i^N \Phi^{-1}[F_i(x_i)]\}}{\phi\{\Phi^{-1}[F_i(x_i)]\} / f_i(x_i)} \tag{9}$$

where F_i and f_i are the cumulative distribution function and the probability density function of the physical variable X_i, respectively. By substituting Eq. (9) into Eq. (3), it can be seen that β is computed by nonlinear optimization in the original space. In practice, this can be easily done using the SOLVER function in EXCEL. Direct application of SOLVER to this example using the mean vector as the starting point will fail. The reason is that variables in original space are not properly scaled. The values of V are about three orders of magnitude larger than the values of h. The correct answer β = 2.60 can only be obtained if one applies "Use Automatic Scaling" under SOLVER Options. The solution in the original space is clearly not robust.

In the Hermite polynomial method, FORM iterations can take place fully in standard Gaussian space. The standard Gaussian variates at each iteration are converted to the original variates using Eq. (5). Step-by-step details are described in Fig. 4. Results for various Hermite expansions are summarized in Table 4 (simulation sample size for Hermite coefficients n = 100). Four-term Hermite expansions for all random variables are sufficient in this example. This example demonstrates that representing non-Gaussian random variables using Hermite expansions in FORM is very robust from a computational viewpoint. It is not necessary to "Use Automatic Scaling" in SOLVER.



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	FIRST-ORDER RELIABILITY METHOD														
2	RANDOM											DETERMINISTIC			
3	Variable	Units	Distribution	mean	cov				λ	ξ		Variable			
4	h	m	Lognormal	2	0.5				0.581575	0.472381		B	m	3	
5	ϕ	degrees	Lognormal	35	0.08				3.552158	0.079872		L	m	3	
6	G	MPa	Lognormal	20	0.5				2.88416	0.472381		γ	kN/m^3	17.7	
7	V	kN	Normal	1500	0.05				1500	75		γsat	kN/m^3	20.3	
8	H	kN	Extreme I	150	0.2				163.5015	23.3909					
9															
10					0	1	2	3	4	5	6	7	8	9	10
11			U		H0	H1	H2	H3	H4	H5	H6	H7	H8	H9	H10
12	U_1		-0.940158		1	-0.94016	-0.1161	1.989471	-1.52211	-6.52686	13.74682	26.23699	-120.8947	-96.2358	1178.52
13	U_2		-2.2249065		1	-2.22491	3.950209	-4.33903	-2.19669	22.24355	-38.5064	-47.78819	375.8689	-453.9678	-2372.78
14	U_3		-4.633E-08		1	-4.6E-08	-1	1.39E-07	3	-6.9E-07	-15	4.86E-06	105	-4.38E-05	-94
15	U_4		0.19110904		1	0.191109	0	0	0	0	0	0	0	0	0
16	U_5		0.29230222		1	0.292302	-0.91456	-0.85193	2.494657	4.136922	-11.2641	-28.11404	70.63056	245.5578	-563.89
17															
18	Variable	Units	X		a0	a1	a2	a3	a4	a5	a6	a7	a8	a9	a10
19	h	m	1.111775		2	0.944761	0.223144	0.035136	0.004149	0.000392	3.09E-05	2.08E-06	1.22E-07	5.95E-09	2.73E-1
20	ϕ	degrees	28.7801951		35	2.795535	0.111643	0.002972	5.94E-05	9.48E-07	1.26E-08	1.43E-10	-3.28E-12	-3.99E-14	-1.51E-1
21	G	MPa	19.9999995		20	9.447614	2.231435	0.351362	0.041494	0.00392	0.000309	2.08E-05	1.22E-06	5.95E-08	2.73E-0
22	V	kN	1514.33318		1500	75	0	0	0	0	0	0	0	0	0
23	H	kN	158.501758		149.9997	29.08653	-5.06576	0.687044	-0.0369	-0.00482	0.0011	0.000523	-9.29E-05	2.86E-05	1.51E-07
24															
25	ϕ	rad	0.50230916												
26	N_q		16.0455798		P		5.45E-07								
27	N_γ		18.726431												
28	γ'	kN/m3	10.49		β		2.440506								
29	γ*	kN/m3	13.1619659												
30					p_f		0.007333								
31	ζ_{ys}		0.6												
32	$\zeta_{\gamma i}$		0.75850883												
33	σ_a'		23.7508977												
34	Δ		0.00096309												
35	I_r		1532.9808												
36	I_{rr}		619.037427												
37	I_{rc}		61.8084011												

Cell	EXCEL functions
C19:C23	Compute X_i from U_i using Hermite polynomials: C19=MMULT(E19:O19,TRANSPOSE(E12:O12)), C20=MMULT(E20:O20,TRANSPOSE(E13:O13)), etc.
G26	Compute performance function, P: G26 = (0.5*$N$4*C29*C27*C31*C32*C38)*$N$4*$N$5-C22
G28	Compute reliability index: G28=SQRT(MMULT(TRANSPOSE(C12:C16),C12:C16))
G30	Compute probability of failure: = NORMSDIST(-G28)
	Solver Set Target Cell: $G$28 Equal To: Min By Changing Cells: $C$12:$C$16 Subject to the Constraints: $G$26 <= 0

Fig. 4 EXCEL implementation of first-order reliability method for footing problem.

Table 4. FORM solutions using Hermite expansions of various lengths.

No. of Hermite terms	Reliability index	Probability of failure
2	2.441	0.00733
3	2.572	0.00505
4	2.600	0.00466
6	2.599	0.00467
Equivalent Gaussian	2.599	0.00467

3. Collocation-based Stochastic Response Surface Method

The first-order reliability method (FORM) is capable of handling any nonlinear performance function and any combination of correlated non-normal random variables. Its accuracy depends on two main factors: (a) the curvature of the performance function at the design point and (b) the number of design points. If the curvature is significant, the second-order reliability method (SORM) (Breitung 1984) or importance sampling (Rackwitz 2001) can be applied to improve the FORM solution. Both methods are relatively easy to implement, although they are more costly than FORM. If there are numerous design points, FORM can underestimate the probability of failure significantly. At present, no solution method exists that is of comparable simplicity to FORM. Note that problems containing multiple failure modes are likely to produce more than one design point. For problems containing a few failure modes that can be individually analyzed by FORM, simple first-order and second-order probability bounds exist (Phoon, 2008a). A more general approach that is gaining wider attention is the spectral stochastic finite element method originally proposed by Ghanem & Spanos (1991). The key element of this approach is the expansion of the unknown random output vector using multi-dimensional Hermite polynomials as basis functions (also called a polynomial chaos expansion).

3.1 Multi-dimensional Hermite Polynomials

The multi-dimensional Hermite polynomials are significantly more complex than the one-dimensional version in Section 2.1. In the case of model inputs with more than one random component, model output can be represented by multi-dimensional Hermite polynomials:

$$Y = a_o + \sum_{i_1=1}^{n} a_{i_1}\Gamma_1(U_{i_1}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(U_{i_1}, U_{i_2}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(U_{i_1}, U_{i_2}, U_{i_3}) + ... \qquad (10)$$

where Y is the output and $\Gamma_p(U_{i_1},...,U_{i_p})$ are multi-dimensional Hermite polynomials of degree p given by:

$$\Gamma_p(U_{i_1},...,U_{i_p}) = (-1)^p e^{\frac{1}{2}U^T U} \frac{\partial^p}{\partial U_{i_1}...\partial U_{i_p}} e^{-\frac{1}{2}U^T U} \qquad (11)$$

where $\{U_{i_k}\}_{k=1}^{p}$ is a vector of independent standard normal variables, n is the number of standard normal random variables used to represent the uncertainty in the model inputs, and $a_0, a_{i_1}, a_{i_1 i_2}$, etc. are the unknown coefficients to be estimated.

One-dimensional Hermite polynomials can be generated easily and efficiently using a 3-term recurrence relation [Eq. (1)]. No such simple relation is available for multi-dimensional Hermite polynomials. They are usually generated using symbolic algebra, which is possibly out of reach of the general practitioner.

Liang et al. (2007) have developed a user-friendly EXCEL add-in to generate tedious multi-dimensional Hermite expansions automatically. This EXCEL add-in is freely available at http://www.eng.nus.edu.sg/civil/people/cvepkk/prob_lib.html.

Once the multi-dimensional Hermite expansions are established, their coefficients can be calculated following the steps described in Section 2.1. The only important practical detail here is that the number of collocation points used to evaluate the coefficients in Eq. (10) should be minimized, because the output at each collocation point can require costly finite element calculations. Phoon and Huang (2007) demonstrated that the collocation points are best sited at the roots of the Hermite polynomial that is one order higher than that of the Hermite expansion. For example, coefficients of a third-order expansion will require collocation points sited at the roots of the fourth-order Hermite polynomial, which are $\pm\sqrt{(3 \pm\sqrt{6})}$. Although zero is not one of the roots, it should be included because the standard normal probability density function is highest at the origin. If there are only two random inputs in the problem, twenty-five collocation points can be generated by combining the roots and zero in two dimensions as illustrated in Fig. 5. The roots of Hermite polynomials (up to order 15) can be calculated numerically as shown in Appendix A.8 of Phoon (2008a).

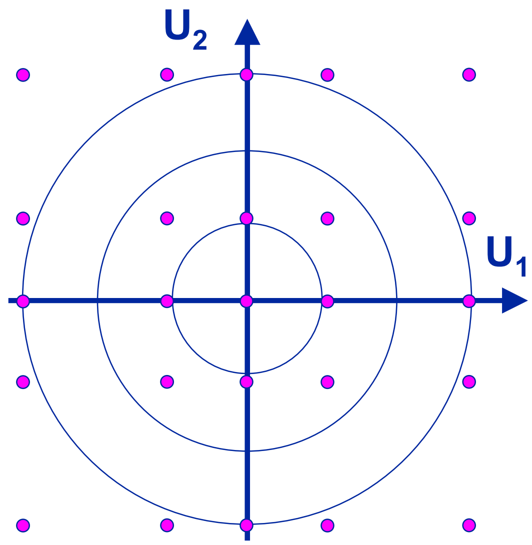


Fig. 5. Collocation points for a third-order expansion with two random dimensions.

3.2 Application to Laterally Loaded Piles

The force system of a laterally loaded drilled shaft is complex and three-dimensional. Although passive lateral soil resistance dominates, there are shearing and axial force components at the tip and along the front, back, and side faces of the shaft. The customary approach is to simplify this actual force system to a two-dimensional distribution of lateral soil resistance (Fig. 6). The ultimate lateral capacity (H_u) and the depth of rotation (z_r) can be calculated based force and moment equilibrium once this simplification is made.

This calculation is relatively easy to perform using Visual Basic, which is freely available in EXCEL (Fig. 7).

The performance function is:

$$P = MH_u - Q \qquad (12)$$

where H_u is a function of the undrained shear strength (s_u), M is the model factor that accounts for idealizations in the evaluation of the lateral capacity, and Q is the variable horizontal load. There are three independent random variables in this problem (s_u, Q, and M). They are assumed to be lognormally distributed. The statistics are: (a) mean s_u = 50 kPa, cov s_u = 50%, (b) mean Q = 200 kN, cov Q = 15%, and (c) mean M = 1.32, cov M = 29% (Phoon and Kulhawy, 2005). The deterministic parameters are: diameter (B) = 1 m, depth/diameter (D/B) = 6.3 and eccentricity (e) = 0.5 m. The mean factor of safety is 4.8.

The output (taken as the performance function P) can be expanded as a third-order expansion:

$$\begin{aligned} P \approx\ & a_o + a_1U_1 + a_2U_2 + a_3U_3 + a_4\left(U_1^2 - 1\right) + a_5\left(U_2^2 - 1\right) + a_6\left(U_3^2 - 1\right) + a_7U_1U_2 + \\ & a_8U_1U_3 + a_9U_2U_3 + a_{10}\left(U_1^3 - 3U_1\right) + a_{11}\left(U_2^3 - 3U_2\right) + a_{12}\left(U_3^3 - 3U_3\right) + \\ & a_{13}\left(U_1U_2^2 - U_1\right) + a_{14}\left(U_1U_3^2 - U_1\right) + a_{15}\left(U_2U_1^2 - U_2\right) + a_{16}\left(U_2U_3^2 - U_2\right) + \\ & a_{17}\left(U_3U_1^2 - U_3\right) + a_{18}\left(U_3U_2^2 - U_3\right) + a_{19}U_1U_2U_3 \end{aligned} \qquad (13)$$

Because s_u, Q, and M are assumed to be lognormals, the equivalent normal mean (λ) and equivalent normal standard deviation (ξ) can be computed and entered as Parameter A and Parameter B in Fig. 8b, respectively. The physical variables are related to the independent standard normal variables as follows:

$$\begin{aligned} s_u &= \exp(0.472U_1 + 3.800) \\ Q &= \exp(0.149U_2 + 5.287) \\ M &= \exp(0.282U_3 + 0.238) \end{aligned} \qquad (14)$$

The key steps involved in the CSRSM EXCEL add-in are quite simple:

1. Define the number of random inputs = 3, the order of the expansion = 3, and the number of outputs = 1 using the “Collo” function at the add-in toolbar (Fig. 8a).
2. Define the probabilistic characteristics of the lognormal random variables (Fig. 8b).
3. Define the number of collocation points = 125 (Fig. 8c).
4. Collocation points are automatically generated (Fig. 8d).
5. The user needs to populate the output column (E) using the input columns (B, C, D) (Fig. 8d). Column E is obtained using the Visual Basic code in Fig. 7. The coefficients of the Hermite polynomials in Eq. (13 are obtained using the “Coeff” function in the add-in toolbar.
6. Once Eq. (13) is fully defined, it is possible to simulate the entire cumulative distribution function of P cheaply. This is carried out using the “Distrb” function in the add-in toolbar.

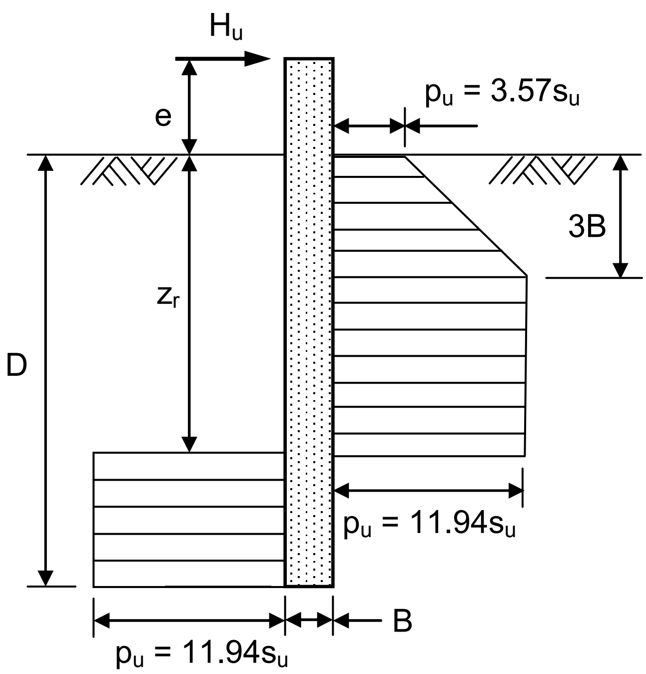


Fig. 6. Calculation of ultimate lateral capacity in clay using lateral soil resistance (p_u) proposed by Randolph & Houlsby (1984)

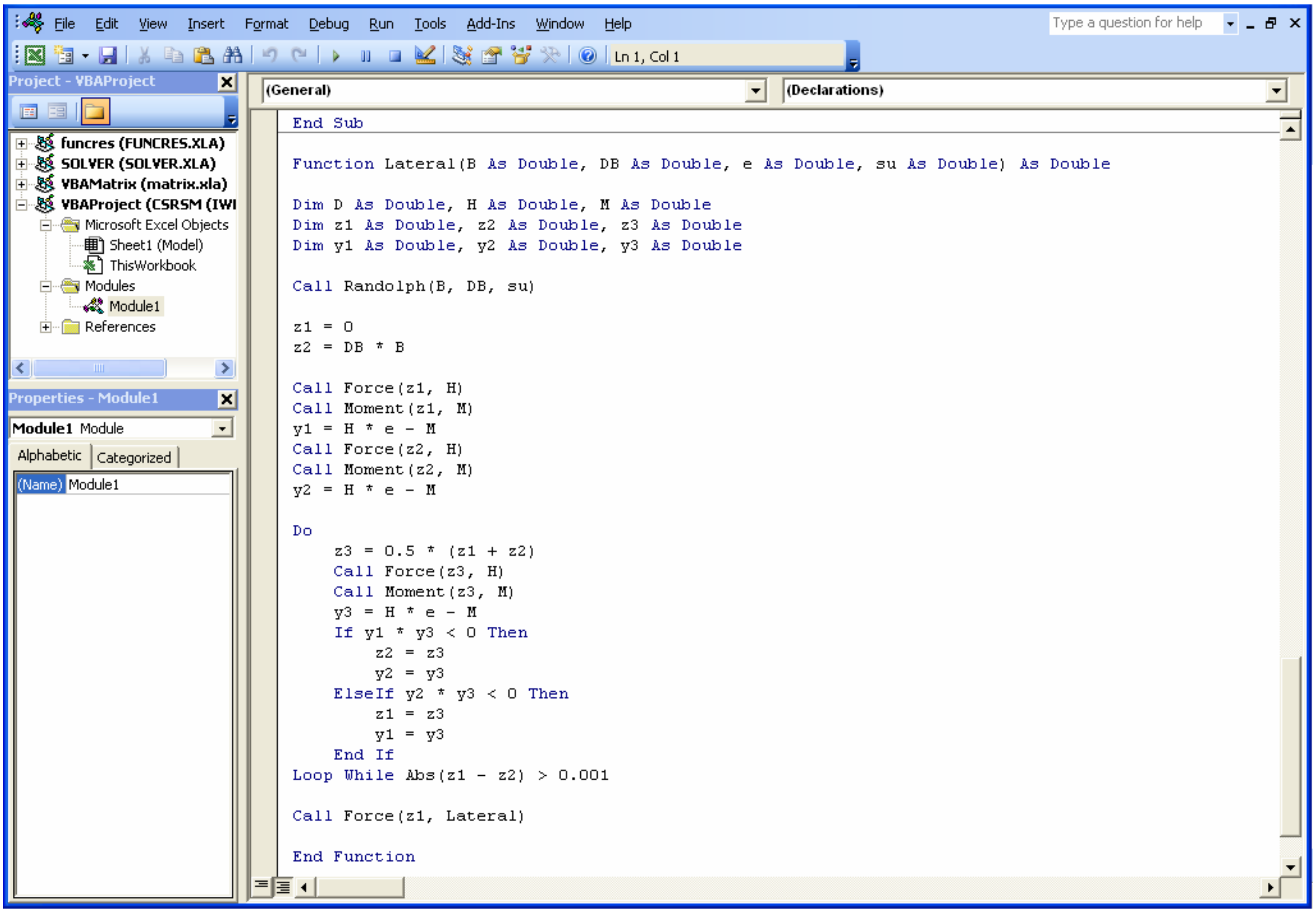
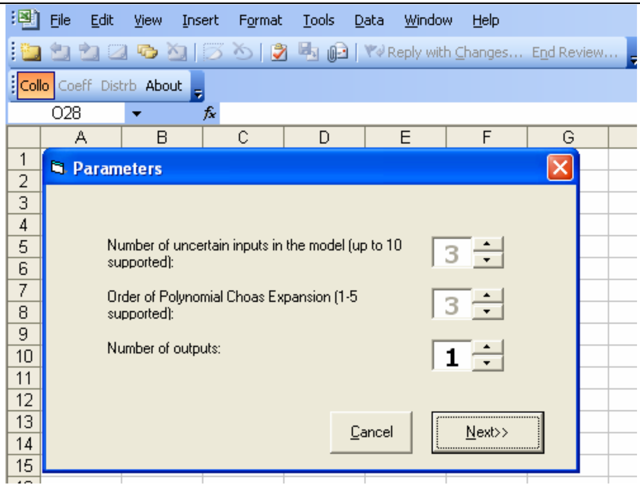
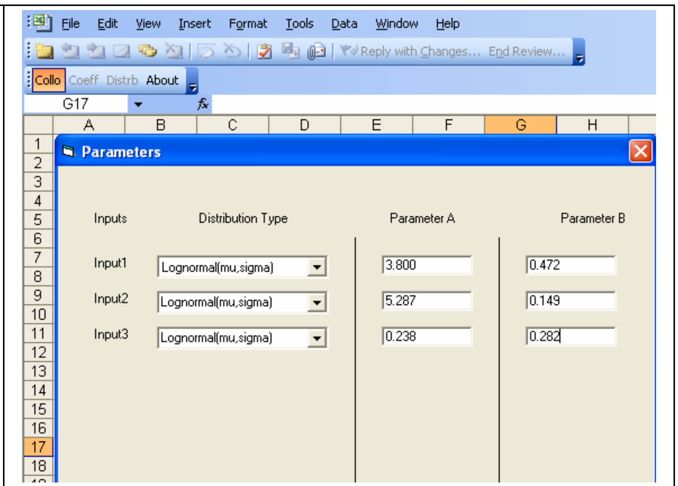


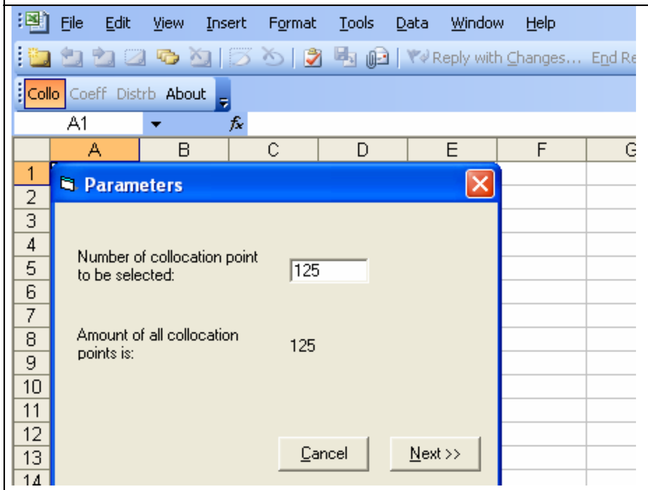
Fig. 7. Visual Basic code in EXCEL to compute H_u.



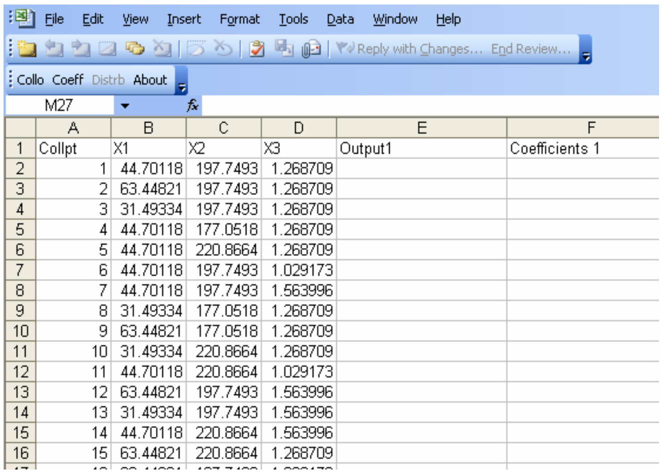
(a) Define number of random inputs, order of expansion, and number of outputs



(b) Define the probabilistic characteristics of the variables



(c) Define the number of collocation points



(d) Generate collocation points in original space

Fig. 8. "Collo" function in CSRSM add-in

The estimated probability of failure based on 125 collocation points and 10,000 realizations is 0.0012. Direct simulation from 10,000 realizations produces 0.0012. The solution from FORM is 0.0014. FORM is able to compute the probability of failure for each failure mode accurately and efficiently. In fact, FORM is more accurate for low probability events! Both CSRSM and simulation reduce in accuracy as probability decreases if all factors are kept constant. The key practical limitation is that FORM cannot handle multiple correlated failure modes in a general way. Although not illustrated herein, CSRSM can handle vector outputs readily. It is worthy to note that CSRSM only requires 125 function calls (each function call can be very expensive for finite element analysis) while direct simulation requires 10000 function calls. Currently, CSRSM suffers from the curse of dimensionality, although progress is being made to mitigate this significant practical problem.

It is worthy to note that the execution of the EXCEL add-in as summarized in Fig. 8 is almost identical to a routine parametric study. The only difference is that the practitioner does not have to struggle with the tabulation of alternate sets of input parameters for the parametric study – this is automatically given by the collocation points as shown in Fig. 8d. In a standard parametric study, input parameters are typically varied based on engineering judgment whereas the collocation points incorporate important statistical information (distributions, correlation structure) systematically. The purpose of a standard parametric study is to gain a "feel" for the sensitivity of the outputs to variations in the input parameters and development of potential mechanisms leading to unsatisfactory system performance. CSRSM will yield more quantitative information such as the variance of the outputs and probability of component/system failure.

4. Conclusions

The objective of this paper is to briefly describe two user-friendly reliability techniques that could be readily implemented in EXCEL. "User-friendly" methods refer to those that can be implemented on a modest desktop PC by a non-specialist with limited programming skills; in other words, methods within reach of the general practitioner.

Non-Gaussian random variables can be represented using Hermite polynomials in a completely general way, even if the data do not fit any known classical probability distributions. This generality is useful for practitioners with limited knowledge of available probability distributions. The First-Order Reliability Method (FORM) is shown to be computationally simpler and more robust when used in conjunction with Hermite polynomials than the commonly used equivalent Gaussian technique. Extension to correlated random vectors is quite straightforward. The key practical limitation is that FORM cannot handle multiple correlated failure modes in a general way.

The collocation-based stochastic response surface method (CSRSM) can handle vector outputs readily. An example is presented to demonstrate that small probabilities of failure can be estimated with minimal function calls using CSRSM. There are two practical bottlenecks. First, the multi-dimensional Hermite expansions are generated using symbolic algebra, which is possibly out of reach of most practitioners. A user-friendly EXCEL add-in to generate tedious multi-dimensional Hermite expansions automatically is introduced in this paper (download: http://www.eng.nus.edu.sg/civil/people/cvepkk/prob_lib.html). Once the multi-dimensional Hermite expansions are established, their coefficients can be calculated following routine steps similar to linear regression. The second practical bottleneck is the curse of dimensionality. Research is on-going to resolve this important problem.

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