Modal Model Reduction for Vibration Control of Flexible Rotor Supported by Active Magnetic Bearing

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Key words: Model reduction, vibration control, flexible rotor, magnetic bearing

ABSTRACT

This paper proposes a criterion to select the modes for modal truncated model of flexible rotor only supported by active magnetic bearings. The proposed approach relies on the concepts of minimum control input and output energy assuming that the system is subjected to transient disturbances. Accurate large order model for the levitated rotor is taken by finite element analysis and transformed to the modal equation. By proposed methodology, which modal states should be retained in the truncated model are investigated over the whole operational speed range by the calculation. Finally, the effectiveness is verified by checking the model error between original model and reduced model.

1. INTRODUCTION

For last few decades, the demands of light weight and higher speed toward higher power density of rotating machinery have been increased so that the rotor bearing system becomes continuously more flexible. The vibration problem of the flexible rotor caused by self-exited force, external disturbances, etc have increasingly attracted the intention of many researchers and the vibration control accordingly became the important issues to operate the machine safely and effectively.

Generally, the accurate dynamic model of flexible rotor system is obtained by finite element analysis [1]; however its high degree of freedom makes the model-based controller design difficult so that numerous model reduction techniques have been developed to derive the smaller order of nominal model for effective approximation.

Among various reduction methods, classic modal truncation is widely adopted mainly because it preserves the system's modal properties that are retained in the model exactly [2]. However the criteria of choosing the 'important' modes for control has not established yet so that it is not always clear which modes should be retained, especially to flexible rotor-bearing system. Modal truncation does not take input-output properties into account either.

Two approaches including the balanced model reduction [3, 4] and Hankel norm approximation [5] are well known for model reduction to be most effective as the input-output properties are considered. The former method is based on transforming the state to a coordinate system in which the controllability and observability gramians are equal and diagonal, and deleting the states having small Hankel singular value which correspondingly means small controllability and observablility gramian. The latter seeks the optimal solution that minimizes the Hankel norm of the differences between nominal and reduced model. If we consider a flexible structure that is described by modal coordinates, then it has been shown [6] that its gramians are approximately diagonal and equal, i.e. modal coordinates are approximately balanced, as long as all modes are lightly damped and all natural frequencies are widely separated. However these two methods are inadequate for reduction of flexible rotor-bearing FEM model because those produce an approximating model over the entire frequency range and lose the physical interpretation which is capable for modal truncation.

In this study, the criteria by which the important modes be selected is proposed based on the initial modal model considering input-output properties of flexible rotor supported two active magnetic bearing system. It is accomplished by comparing the minimum input energy and output energy of each mode over the whole operational speed. For the control implementation subjected to transient disturbances, small order modal state model is obtained and the model error verification between original and reduced one follows.

2. FE Analysis and Modal Equation

Finite element model of a flexible rotor system supported by active magnetic bearing is shown in figure 1; it consists of 21 elements and is modeled by Rayleigh beam elements with isotropic bearing stiffness and

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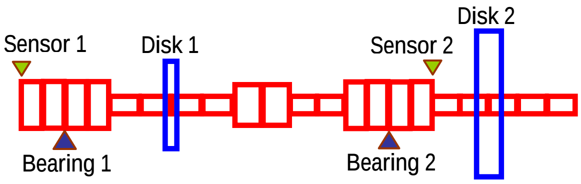

Figure 1. 21 elements FEM model

damping of $33\times10^4\ N/m$ and $200\,Ns/m$ respectively [1]. Through FE analysis, 44 dof equation of motion in complex domain can be written as following

$$\mathbf{M}\ddot{\mathbf{p}}+\left(\mathbf{C}-j\Omega\mathbf{G}\right)\dot{\mathbf{p}}+\mathbf{K}\mathbf{p}=\mathbf{g} \qquad (1)$$

where $\mathbf{M},\mathbf{C},\mathbf{G}$ and $\mathbf{K}$ are the mass, damping, gyroscopic and stiffness matrices; $\mathbf{p}=\mathbf{y}+j\mathbf{z}$ is the complex displacement vector of each nodes and $\mathbf{g}$ is external forcing vectors. By means of the modal transformation and the modal expansion, Modal equation of 88 dof can be obtained as

$$\begin{aligned}\dot{\boldsymbol{\zeta}}&=\mathbf{A}\boldsymbol{\zeta}+\mathbf{B}\mathbf{g}\\ \mathbf{y}&=\mathbf{C}\boldsymbol{\zeta}\end{aligned} \qquad (2)$$

where

$$\begin{aligned}\mathbf{A}&=diag\left\{\lambda_1^B \quad \lambda_1^F \quad \lambda_2^B \quad \cdots \quad \lambda_{44}^F\right\}\\ \mathbf{B}&=\begin{bmatrix} {}_3u_1^B & {}_3u_1^F & {}_3u_2^B & \cdots & {}_3u_{44}^F \\ {}_{15}u_1^B & {}_{15}u_1^F & {}_{15}u_2^B & \cdots & {}_{15}u_{44}^F\end{bmatrix}^T\\ \mathbf{C}&=\begin{bmatrix} {}_1u_1^B & {}_1u_1^F & {}_1u_2^B & \cdots & {}_1u_{44}^F \\ {}_{17}u_1^B & {}_{17}u_1^F & {}_{17}u_2^B & \cdots & {}_{17}u_{44}^F\end{bmatrix}\end{aligned} \qquad (3)$$

and $\boldsymbol{\zeta}=\left\{\zeta_1^B \quad \zeta_1^F \quad \zeta_2^B \quad \cdots \quad \zeta_{44}^F\right\}^T$ is the complex modal state vector. $\mathbf{g}=\left\{g_3 \quad g_{15}\right\}^T$ denotes actuator force vector at third and fifth nodes and $\mathbf{y}=\left\{y_1 \quad y_{17}\right\}^T$ is the sensor output vector at first and 17^{th} nodes. In the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, λ and u is the eigenvalue and modal input coefficient respectively and the eigenvalue is denoted by $\lambda=\sigma+i\omega$. Upper right superscripts, B and F, mean backward and Forward direction. Lower right subscript means corresponding modal equation number. Lower left subscript means corresponding forcing node.
The whirl-speed chart is shown in figure 2 which just illustrates 10 modes. Due to gyroscopic effect, it is found that the natural frequencies are varied with respect to rotational speed.

3. Modal Model Reduction

Suppose that the whole rotor system is perturbed by two

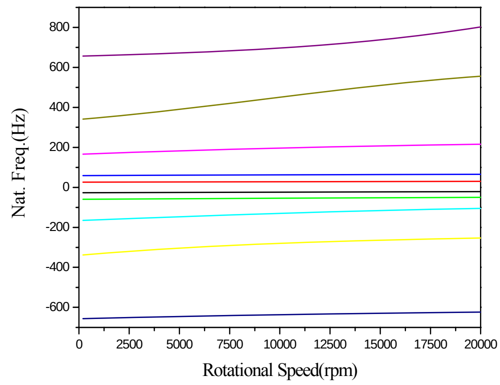

Figure 2. Whirl-speed Chart

equal impulses at bearings' nodes so that the initial condition $\boldsymbol{\zeta}(0)=\boldsymbol{\zeta}_0$ holds. Minimum control energy required to reach the desired modal state $\boldsymbol{\zeta}(T)=\boldsymbol{\zeta}_T=0$ can be computed by solving minimum control energy problem with the constraints of initial conditions as following [7, 8]

$$\text{minimize } \ E(\mathbf{g})=\int_0^T \mathbf{g}^H(t)\mathbf{g}(t)dt \qquad (4)$$

where superscript H menas conjugate transpose of the matrix. The optimal solution is given by

$$\mathbf{g}_o(t)=-\mathbf{B}^H e^{\mathbf{A}^H(T-t)}\mathbf{P}^{-1}(T)\left(e^{\mathbf{A}T}\boldsymbol{\zeta}_0-\boldsymbol{\zeta}_T\right) \qquad (5)$$

where $\mathbf{P}(t)$ is the controllability gramian matrix defined by

$$\begin{aligned}\mathbf{P}(t)&=\int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^H e^{\mathbf{A}^H\tau}d\tau\\ &=\int_0^t diag\left(e^{\lambda_r^i t}\right)\mathbf{B}\mathbf{B}^H diag\left(e^{\bar{\lambda}_s^k t}\right)d\tau \qquad (6)\\ i,k&=B,F \text{ and } r,s=1,2,3,\ldots,44\end{aligned}$$

Each elements of the controllability gramian matrix $\mathbf{P}(t)$ is expressed like

$$\begin{aligned}P_{rs}^{ik}(t)&=\int_0^t e^{\lambda_r^i t}\mathbf{b}_r^i\mathbf{b}_s^{k\,H}e^{\bar{\lambda}_s^k t}d\tau\\ &=\frac{{}_3u_r^i\cdot{}_3\bar{u}_s^k+{}_{15}u_r^i\cdot{}_{15}\bar{u}_s^k}{-\left(\lambda_r^i+\bar{\lambda}_s^k\right)}\left(1-e^{\left(\lambda_r^i+\bar{\lambda}_s^k\right)t}\right)\end{aligned} \qquad (7)$$

By applying control law of equation (5), the minimum control energy index $\mathbf{E}_g$ becomes

$$\mathbf{E}_g=\left(e^{\mathbf{A}T}\boldsymbol{\zeta}_0-\boldsymbol{\zeta}_T\right)^H\mathbf{P}^{-1}(T)\left(e^{\mathbf{A}T}\boldsymbol{\zeta}_0-\boldsymbol{\zeta}_T\right) \qquad (8)$$

At this point, it should be pointed out that the controllability gramian is approximately diagonal in modal equation [6]. Therefore $\mathbf{E}_g$ is also diagonal

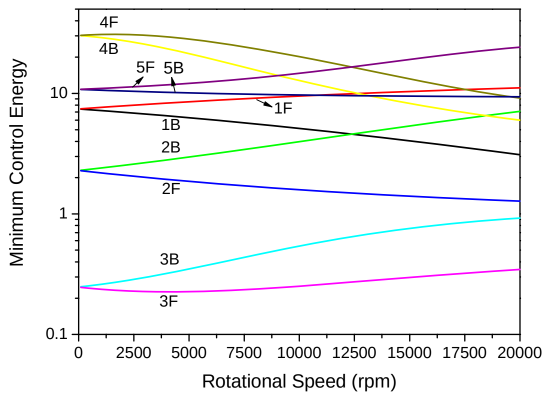

Figure 3. Minimum Control Energy

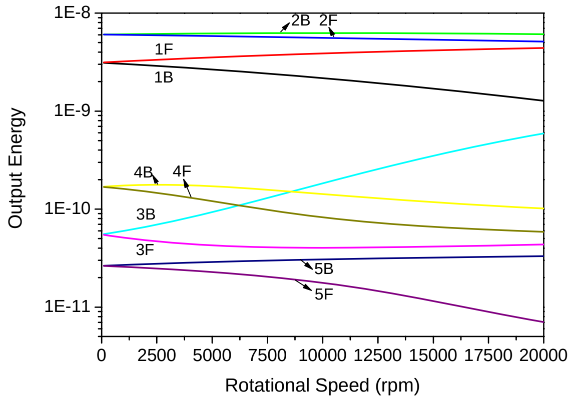

Figure 4. Output Energy

dominant matrix so that it can be very closely approximated by the summation of diagonal elements of which each means the minimum control energy of corresponding mode.

$$\mathbf{E}_g \cong \sum_{i=B,F} \sum_{r=1}^{22} E_{r0}^{i}$$

$$E_{rg}^{i} = \frac{2e^{2\sigma_r^i T}\left(e^{2\sigma_r^i T}-1\right)\sigma_r^i \left|{}_3u_r^i + {}_{15}u_r^i\right|^2}{\left\|\mathbf{b}_r^i\right\|_2} \qquad \text{(9-a,b)}$$

Figure 3 shows minimum control energy E_{rg}^{i} in case of $T = 0.05s$.

Next, assuming the system is on the initial condition $\zeta(0) = \zeta_0$ with $\mathbf{g}(t) = \mathbf{0}, t \geq 0$, then the output energy is [5]

$$\mathbf{E}_y = \int_0^\infty \mathbf{y}^H(t)\mathbf{y}(t)dt = \zeta_0{}^H \mathbf{Q} \zeta_0 \qquad (10)$$

where $\mathbf{Q}$ is the controllability gramian matrix defined by

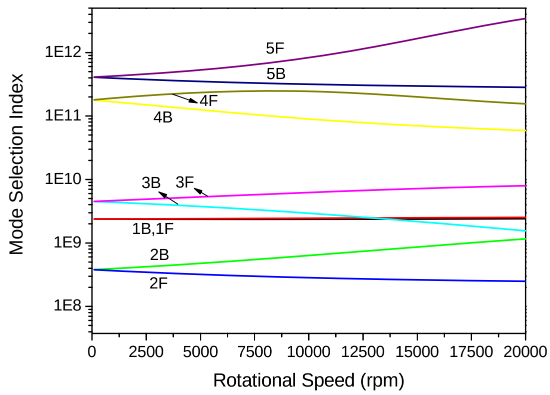

Figure 5. Mode Selection Index

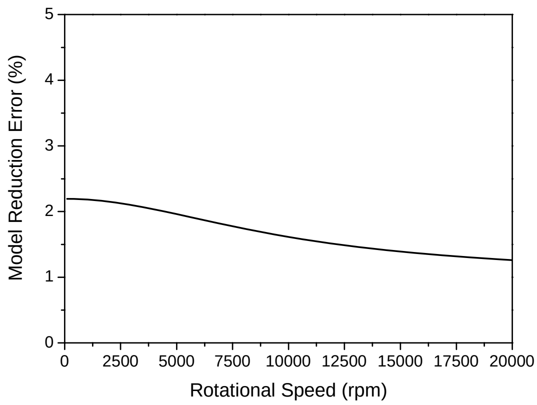

Figure 6. Model reduction error

$$\mathbf{Q} = \int_0^\infty diag\left(e^{\lambda_r^i t}\right)\mathbf{C}\mathbf{C}^H diag\left(e^{\bar{\lambda}_s^k t}\right)d\tau \qquad (11)$$

where $\mathbf{Q}$ also is diagonal dominant matrix, accordingly the diagonal elements is expressed like

As a result, equation (10) described by equation (12) becomes

$$\mathbf{E}_y \cong \sum_{i=B,F} \sum_{r=1}^{44} E_{ry}^{i}$$

$$E_{ry}^{i} = \frac{\left\|\mathbf{c}_r^i\right\|_2 \left|{}_3u_r^i + {}_{15}u_r^i\right|^2}{-2\sigma_r^i} \qquad \text{(13-a,b)}$$

By comparing the ratio between equation (9-b) and (13-b) for each mode, mode selection index can be defined as

$$MSI_r^i = \frac{E_{rg}^i}{E_{ry}^i} = \frac{4e^{2\sigma_r^i T}\left(1-e^{2\sigma_r^i T}\right)\sigma_r^{i\,2}}{\left\|\mathbf{b}_r^i\right\|_2 \left\|\mathbf{c}_r^i\right\|_2} \qquad (14)$$

In figure 5 is shown a plot of mode selection index of equation (14) versus the rotational speed and first five modes (1B, 1F, 2B, 2F and 3B) of which MSI value is small are chosen for modal truncation.
Model reduction error for five modes truncation can be expressed as a percentage by

$$MRE = \frac{\sum_{i=B,F} \sum_{r=1}^{44} E_{ry}^{i} - \sum_{i=B,F} \sum_{r=1}^{n} E_{ry}^{i}}{\sum_{i=B,F} \sum_{r=1}^{44} E_{ry}^{i}} \times 100 \qquad (15)$$

This model reduction error is shown in figure 6, which illustrated the error can be considered less than two percent for the most of the operational speed.

4. Conclusion

In this paper, a mode selection criterion called as Mode Selection Index (MSI) is proposed for the modal truncation of flexible rotor system supported by two active magnetic bearings. On the assumption of impulse disturbances at bearing locations, MSI is determined by the ratio between minimum input energy and output energy. Using MSI, five lower modes are chosen for reduction of which error are found to be around two percent.

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