제 3 자 물류 허브 창고의 생산납기구간 수요에 대한 인바운드 선적계획
Inbound Shipment Planning for Dynamic Demands with Production Time Windows at A Third-Party Warehouse Hub

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Abstract

This paper considers a shipment planning of products from manufacturers to a third-party warehouse for demands with production time windows where a demand must be replenished in its time window. The underling lot-sizing model also assumes cargo delivery cost in each inbound replenishment to the warehouse. For this model, an optimal $O(nT^4)$ is presented where n is the number of demands and T is the length of the planning horizon.

Key words: dynamic lot-sizing, inventory/ production, shipment consolidation, time windows, logistics

1. Introduction

This paper considers a supply system inbound to a main third-party warehouse (called *hub*) for demands with customer specific *time windows*. An areal warehouse (called *center*) gathers semi-finished products from its neighbor manufacturers, providing additional labeling and packaging operations to make final goods, and then delivers them inbound to the third-party warehouse hub for dispatching to end customers. This services for production firms, including the final labeling and packaging operations, are provided by a third-party logistics provider who has experties in delivery and transportation. As the specification of each product (in a demand) is chiefly characterized by the type of raw material (semi-product) from manufacturers, the replenishment of each product in the center highly depends on the availability of semi-products. Hence, each demand cannot be satisfied until the period when the appropriate semi-product is available in the center. Along with this first available period (*earliest due period*), each demand also has *latest due period* by which it should be fulfilled. Then the two periods, earliest due period and latest due period, defines the time window of a demand. Given a demand i with time window $[E_i, L_i]$, it should be replenished during the time window and be dispatched at the period L_i. This type of time window is called *production time window* (Dauzère-Pérès 2002).

To provide a cost-effective schedule over periods, this paper studies a lot-sizing model for dynamic demands with production time windows. Major cost components include replenishment cost for labeling and packaging operations in the center, product storage cost in the hub warehouse and delivery cost between the two warehouses center and hub. Final goods are shipped by *cargos*, which is delivered to the main warehouse hub by transportation fleets, like, trucks and railroads. The transportation operation is costed by the number of cargos treated, which motivates *shipment consolidation program* to increase *full-truck-load* (FTL) cargos while decreasing *less-than-truck-load* (LTL) cargos in the areal warehouse center. Under the situation that demands are given by production time windows, the chance of product consolidation might increase because of the time window flexibility. The literature has a few results on the models for time windows or for shipment planning; however, to the best knowledge of the author, until now, no research has been done for the integrated model jointly considering the cargo capacity and demands with time windows.

The literature is reviewed in the next Section and the model is formulated in Section 3. Optimality properties are developed in Section 4 and an $O(nT^4)$ algorithm is designed in Sections 4-6. Future research is discussed in the final Section.

2. Relavant Literature

The single-item dynamic lot-sizing model by Wagner and Whitin (1958) assumes demands with single due dates and instantaneouse deliveries without costs from production facility to warehosue. For this uncapacitated problem, efficient solution procedures are provided by Aggarwal and Park (1993), Federgruen and Tzur (1991), and Wagelmans et al. (1992). When production is limited by facility capacity, the problem is known to have quite different problem nature than the Wagner-Whitin model. For this capacitated lot-sizing models, see Florian and Klein (1971), Bitran and Yanasse (1982), Chung and Lin (1988) and Van Hoesel and Wagelmans (1996).

The researches on time windows in lot-sizing are started by Lee et al. (2001) and Dauzère-Pérès et al. (2002). Lee et al. (2001) considers *delivery time windows*, a kind of grace periods, during which no penalty cost occurs. For this type of time windows, see Jaruphongsa et al. (2004a, b), Hwang and Jaruphongsa (2006, 2008) and Hwang (2007a). The production time window under this study is investigated in Dauzère-Pérès et al. (2002). They also classify lot-sizing problems based on time window structure. When no time window strictly overlays others, the model is called *non-customer specific*. If there is no such restriction, the general model is called *customer specific*. Recently, Brahimi et al. (2006) extended the model into capacitated multi-item case with non-customer/ customer specific production time windows and Wolsey (2006) presents tight extended formulations for various production and delivery time window models. Hwang (2007b) solves the problem for general cost structures. For capacitated lot-sizing problem with both deliver and production time window cases, Hwnag et al. (2007e) provides a $O(nT^4)$ algorithm. As we shall see later, the optimal algorithm in this paper is based on similar decomposition principle in Hwnag et al. (2007e).

The inbound shipment plannning model has originated from the multiple setups model by Lippman (1969), which contains no replenishment costs. Lee (1989) extends the model by treating a replenishment cost. The application of this concept of multiple setups at third-party warehouse is done by Lee et al. (2003) for inventory replenishment and outbound shipment scheduling. Several variants of inbound shipment models have been studied in Lee (2004), Jaruphongsa et al. (2005) and Hwang (2007c, d). However, these models assume that demands are given by single periods and until now no result has come out to treat shipment planning for demands with time windows. This paper deals with an inbound shipment model with customer-specific production time windows. In the following we present the formal mathematical model of our problem.

3. Problem Formulation and Optimality Properties

Let T denote the length of a planning horizon and n the number of demands. For each period $t \in \{1, 2, \ldots, T\}$ and demand $i \in \{1, 2, \ldots, n\}$ we define:

- C: cargo capacity;
- A: unit cargo delivery cost;
- d_i : required quantity of demand i;
- $[E_i, L_i]$:time window of demand i
- r_t: required quantity that should be dispatched for demands with latest due periods of t; $r_t = \sum_{i:L_i=t} d_i$;
- x_t: replenishment level in t;
- I_t: inventory level in t;
- $p_t(x_t)$: replenishment cost function in t for the amount x_t with $p_t(0) = 0$. We assume *nonspeculative* replenishment function such that $p_t(x_t) = K_t + p_t x_t$ for $x_t > 0$ where $p_t \geq p_{t+1}$. Here, K_t and p_t denote the setup and the unit replenishment costs in period t;
- h_t: unit inventory holding cost in t.

We use $\lceil x \rceil$ and $\lfloor x \rfloor$ to denote the smallest integer no less than x and the largest integer no greater than x, respectively. We define the following simple operators in regard with cargo capacity:

- $\bar{n}(x)$: minimum number of cargos to carry x item units; $\bar{n}(x) = \lceil x/C \rceil$;
- $n(x)$: minimum number of FTL cargos to carry x item units; $n(x) = \lfloor x/C \rfloor$;
- $\bar{m}(x)$: number of items that can be carried in $\bar{n}(x)$ cargos; $m(x) = \bar{n}(x) \cdot C$;
- $m(x)$: number of items carried in FTL cargos for the amount of x; $m(x) = n(x) \cdot C$;
- $\Delta(x)$: number of items carried in an LTL cargo for the amount of x; $\Delta(x) = x - m(x)$.

Since items are consolidated and moved by cargo, it costs $\bar{n}(x) \cdot A$ when delivering x item units. The lot-sizing model of minimizing the total costs for demands with production time windows is formulated as:

$$\text{Minimize } \sum_{t=1}^{T} (\bar{n}(x_t) \cdot A + p_t(x_t) + h_t I_t) \qquad (1)$$

Subject to

$$I_{t-1} + x_t - r_t - I_t = 0 \qquad t = 1, \ldots, T, \qquad (2)$$

$$x_t - \sum_{i=1}^{n} y_{it} = 0 \qquad i = 1, \ldots, n, \qquad (3)$$

$$\sum_{t \in [E_i, L_i]} y_{it} = d_i \qquad i = 1, \ldots, n, \qquad (4)$$

$$y_{it} \geq 0 \qquad i = 1, \ldots, n, \; t \in [E_i, L_i], \qquad (5)$$

$$y_{it} = 0 \qquad i = 1, \ldots, n, \; t \notin [E_i, L_i], \qquad (6)$$

$$x_t \geq 0 \qquad t = 1, \ldots, T. \qquad (7)$$

The model (1)–(7) can be transformed into the one with no inventory. Consdier the following objective function

$$\text{Minimize } \sum_{t=1}^{T} (\bar{n}(x_t) \cdot A + p_t(x_t)) \qquad (8)$$

Using results in Wolsey (2007) and Hwang (2007b), we can show that the model (1)–(7) is equivalent to the one (8), (3)–(7), which has no inventory holding costs and no constraints with inventory. From now on, we will focus on the revised model (8), (3)–(7) without inventory in which each demand is satisfied instantaneously at the time of replenishment.

For notational convenience, we let $v_{s,t} = v_s + v_{s+1} + \cdots + v_t$ if $s \leq t$ and $v_{s,t} = 0$ if $s > t$, for any series of values $v_1, v_2, \ldots, v_T$. Then, $x_{s,t}$ and $d_{s,t}$ represent the cumulative sums of replenishments and demands from s through t, respectively. Now, we present notations for periods and demands. About periods, we need to have the following definitions:

- A period t is called a *replenishment* period if $x_t > 0$.
- A replenishment period t is called a *complete* period if any demand with its latest due period no earlier than t is satisfied at or after period t.
- A replenishment period t is called a *full-truck-load* (FTL) period if its replenishment quantity is a multiple of C, i.e., $x_t = m(x_t)$. Any replenishment period, which is not FTL, is referred to as a *less-than-truck-load* (LTL) period.

Without loss of generality, we assume that demands are given by earliest due periods such that demand i precedes demand j if $E_i < E_j$ or $E_i = E_j \leq L_j < L_i$. In regard with demands, we introduce the following definitions: For $1 \leq \lambda < \gamma \leq T+1$,

- $D(\lambda, \gamma)$ denotes a set of demands whose latest due periods are between λ and γ, i.e., $D(\lambda, \gamma) = \{i: \lambda \leq L_i < \gamma\}$;
- $a(\lambda, \gamma)$ denotes the first demand whose latest due period is between periods λ and γ, i.e, $a(\lambda, \gamma) = \min\{i: i \in D(\lambda, \gamma)\}$.

Based on these definitions, we present Properties 1–3 on replenishment periods in relation with set $D(\lambda, \gamma)$ and demand $a(\lambda, \gamma)$.

Property 1. There exists an optimal solution such that if period λ is an LTL replenishment period, then any demand $i \in D(\lambda, T+1)$ is supplied only by replenishments in periods no earlier than λ.

Property 2. There exists an optimal solution such that if demand $i \in D(\lambda, \gamma)$ is supplied by replenishment in period λ, then any demand $j > i$ with $j \in D(\lambda, \gamma)$ is supplied only by replenishments during $\{\lambda, \lambda + 1, \ldots, \gamma - 1\}$.

Property 3. Consider a subproblem of satisfying demands $i \in D(1, \gamma)$ by replenishments during periods $\{1, 2, \ldots, \gamma-1\}$. There exists an optimal subplan such that a complete replenishment period λ exists in which at least one unit of demand $a(\lambda, \gamma)$ is replenished and after which all the replenishments are FTL.

We let $F(\gamma)$ be the minimum cost in satisfying demands $D(1, \gamma)$ for $1 \leq \gamma \leq T$. Then, the optimum cost is $F(T)$. Now, let us define $G(\lambda, \gamma)$, $1 \leq \lambda < \gamma \leq T + 1$, as the minimum cost of satisfying all demands $i \in D(\lambda, \gamma)$ by replenishments during periods $\{\lambda, \lambda + 1, \ldots, \gamma - 1\}$ under the constraints

(i) period λ is a complete replenishment period and each replenishment during $\{\lambda + 1, \ldots, \gamma - 1\}$ is FTL, and
(ii) at least one unit of demand $a(\lambda, \gamma)$ is replenished in period λ.

Then Property 1–3 suggests that the optimal solution $F(T)$ can be found by the following recursion procedure:

$$F(1) = 0,$$
$$F(\gamma) = \min_{1\le\lambda<\gamma}\{F(\lambda)+G(\lambda,\gamma)\}, \qquad \gamma = 2,\ldots,T+1. \tag{9}$$

Note that this procedure is solved in $O(T^2)$ given $G(\lambda, \gamma)$. The next Section presents how to compute $G(\lambda, \gamma)$.

4. Optimal Algorithm

To develop decomposition principle, we introduce a set $D(i \mid \lambda, \gamma)$ and a demand $a(i|\lambda, \gamma)$, for $i \in D(\lambda, \gamma)$ and $1 \le \lambda < \gamma \le T+1$:

- $D(i|\lambda, \gamma)$ denotes the set of all demands $j \ge i$ with $j \in D(\lambda, \gamma)$. Also, we define $D(n+1 \mid \lambda, \gamma) = \varnothing$;
- $C(i|\lambda, \gamma)$ denotes the set of all *crossing* demands $j \ge i, j \in D(i|\lambda, \gamma)$, with $E_j \le \lambda \le L_j < \gamma$. Also, we define $C(n+1 \mid \lambda, \gamma) = \varnothing$;
- $a(i|\lambda, \gamma)$ denotes the smallest demand j with latest due periods between λ and γ. That is $a(i|\lambda, \gamma) = \min\{j : j > i, j \in D(i|\lambda, \gamma)\}$. If the set $\{j : j > i, j \in D(i|\lambda, \gamma)\}$ is empty then we let $a(i \mid \lambda, \gamma) = n+1$.

Then, the set $D(i \mid \lambda, \gamma)$ has the following relationship with demand $a(i \mid \lambda, \gamma)$:

$$D(i \mid \lambda, \gamma) = \{i\} \cup D(a(i|\lambda, \gamma) \mid \lambda, \gamma).$$

We use $d_{i|\lambda,\gamma}$ to denote the total sum of demands in $D(i \mid \lambda, \gamma)$. Then we also have $d_{i|\lambda,\gamma} = d_i + d_{a(i|\lambda,\gamma)|\lambda,\gamma}$. It is not difficult to see that we can compute all values of $a(i \mid \lambda, \gamma)$, $d_{i|\lambda,\gamma}$ in $O(nT^2)$ time. Note that the set $D(\lambda, \gamma)$ is denoted as $D(a(\lambda, \gamma) \mid \lambda, \gamma)$ and its total sum as $d_{a(\lambda,\gamma)|\lambda,\gamma}$.

Let θ be some part of demand i, $0 < \theta \le d_i$. Then, a $(\theta, i|\lambda, \gamma)$-problem is to obtain a minimum schedule over periods $\{\lambda, \lambda+1, \ldots, \gamma-1\}$ such that

(i) every replenishment during $\{\lambda, \lambda+1, \ldots, \gamma-1\}$ is FTL except for the first replenishment period,
(ii) every demand in $D(i|\lambda, \gamma)$ is fulfilled but the demand i can be partially supplied by θ units, $0 < \theta \le d_i$.

Based on this definition, we introduce three costs $g_0(i|\lambda, \gamma)$, $g_1(i|\lambda, \gamma)$ and $f(\theta, i|\lambda, \gamma)$ for $(\theta, i|\lambda, \gamma)$-problems:

- $g_0(i|\lambda, \gamma)$ is the minimum cost of the $(d_i, i|\lambda, \gamma)$-problem,
- $g_1(i|\lambda, \gamma)$ is the minimum cost of the $(d_i, i|\lambda, \gamma)$-problem in which a replenishment occurs in period λ, and
- $f(\theta, i|\lambda, \gamma)$ is the minimum cost of the $(d_i, i|\lambda, \gamma)$-problem in which demand i is supplied from the replenishment in period λ.

Then, we can see that $G(\lambda, \gamma)$ is computed by $f(\cdot)$ as follows:

$$G(\lambda, \gamma) = f(d_{a(\lambda,\gamma)}, a(\lambda, \gamma)|\lambda, \gamma). \tag{10}$$

The remaining task is how to compute $f(\theta, i|\lambda, \gamma)$, which will require the computations of $g_0(i|\lambda, \gamma)$, $g_1(i|\lambda, \gamma)$. To this end, another useful properties are presented.

Property 6. Consider a $(\theta, i|\lambda, \gamma)$-problem such that the first replenishment periord for demand i occurs in period $\tau \in [\lambda, \gamma-1]$. Then there exists an opimal schedule such that if every replenishment during $\{\tau, \tau+1, \ldots, \gamma-1\}$ is FTL, then the total sum of replenishment quantities during $\{\tau, \tau+1, \ldots, \gamma-1\}$ is $m(\theta + d_{a(i|\tau,\gamma)|\tau,\gamma})$.

Suppose that demand i has replenishments in periods $\tau(1), \tau(2), \ldots \tau(k)$ for $E_i \le \tau(1) \le \tau(2) \le \cdots \le \tau(k) \le L_i$. We further suppose that demand i has $\theta_{\tau(j)}$ uints supplied in each period $\tau(j)$ for $j = 1, 2, \ldots, k$. Then a replenishment period $\tau(j)$ with $\theta_{\tau(j)} \ge C$ is called a *major* replenishment period for demand i. Note that each of the periods $\tau(1), \tau(2), \ldots \tau(k)$ is either LTL or FTL period. The following property limists the number of LTL and major periods in a demand.

Property 7. There exists an optimal solution such that LTL or major replenishment for a demand occurs at most once.

Then the next property constrains the positions of LTL and major replenishment periods.

Property 8. Consider a demand i which is replenished in periods $\tau(1), \tau(2), \ldots \tau(k)$ for $E_i \le \tau(1) \le \tau(2) \le \cdots \le \tau(k) \le L_i$. There exists an optimal solution such that

(a) if demand i has an LTL replenishment, it must occur at the first replenishment period $\tau(1)$, and
(b) if demand i has a major replenishment, it must occur at the last replenishment period $\tau(k)$.

5. Computing $f(\theta, i|\lambda, \gamma)$

For notational convenience, we let $a_0 = a(i|\lambda, \gamma)$ and $D_0 = D(a_0|\lambda, \gamma)$. Suppose that every demand D_0 is replenished in period λ as well as the θ units of demand i. Note that this case is possbile only when every demand in $D(i|\lambda, \gamma)$ should be crossing the period λ, i.e., $D(i|\lambda, \gamma) = C(i|\lambda, \gamma)$. We use $f_0(\theta, i|\lambda, \gamma)$ to denote the cost $f(\theta, i|\lambda, \gamma)$ in this case of whole satisfaction in a single period. Then, we have

$$f_0(\theta, i \mid \lambda, \gamma) = \min\begin{cases} K_\lambda + p_\lambda(\theta + D_0), & \text{if } D(i \mid \lambda, \gamma) = C(i \mid \lambda, \gamma), \\ \infty, & \text{otherwise.}\end{cases} \tag{11}$$

Now we consider the other case that at least one replenishment takes place during $\{\lambda+1, \lambda+2, \ldots, \gamma-1\}$. In particular, we chiefly consider two cases:

(i) the θ units of demand i are all satisfied by the replenishment in period λ,
(ii) the θ units of demand i are satisfied not only by period λ but also by some periods during $\{\lambda+1, \ldots, \gamma-1\}$

4.1 Case 1: θ units of demand i are all satisfied by the replenishment in period λ

This case will be treated by considering two subcases

- period λ replenishes only demand i, and
- period λ replenishes not only demand i but also demands D_0.

We first suppose that period λ replenishes only demand i. Recall that every replenishment period during $\{\lambda, \ldots, \gamma-1\}$ should be FTL possibly except for the first period λ. Since demands D_0 are all replenished during $\{\lambda+1, \ldots, \gamma-1\}$, the total quantity D_0 must be a multiple of cargo capacity C. Thus, the assumption that period λ replenishes only demand i leads to $D_0 = m(D_0)$ or $\Delta(D_0) = 0$. Note that the replenishment cost in period λ is $K_\lambda + p_\lambda\theta$. We next consider the cost for demands D_0. Since they are not replenished in period λ, the cost of satisfying them is $g_0(a_0|\lambda, \gamma)$ by definition. Hence, the cost $f(\theta, i|\lambda, \gamma)$ in this case is computed by the following formula:

$$f(\theta, i|\lambda, \gamma) = K_\lambda + p_\lambda\theta + g_0(a_0|\lambda, \gamma).$$

We next suppose that period λ replenishes not only demand i but also demands D_0. Recall that every replenishement during periods $\{\lambda+1, \ldots, \gamma-1\}$ is FTL and θ units of demand i is not replenished during them. This implies that the θ units and the partial quantity $\Delta(D_0) > 0$ should be replenished in period λ. We first consider the cost for demands D_0. It is known that the supply of period λ includes partial amount $\Delta(D_0)$ of demands D_0 but it is not known whether the supply also includes demand a_0. Hence, the cost for D_0 is $g_1(a_0|\lambda, \gamma)$. Consider the replenishement of θ units in period λ. The one-time setup cost K_λ is included in $g_1(a_0|\lambda, \gamma)$; We should not count the setup cost in computing $f(\theta, i|\lambda, \gamma)$ but the cost $p_\lambda\theta$ of replenishing the θ units. Hence, we have

$$f(\theta, i|\lambda, \gamma) = p_\lambda\theta + g_1(a_0|\lambda, \gamma).$$

We use $f_1(\theta, i|\lambda, \gamma)$ to denote the cost of $f(\theta, i|\lambda, \gamma)$ in the case when the θ units of demand i are all satisfied by the replenishment in period λ. Combining the two formulas, we have the following complete formula:

$$f_1(\theta,i\mid\lambda,\gamma) = \min_{\lambda<\tau\le L_i}\begin{cases} K_\lambda + p_{i\lambda}\theta + g_0(a_0\mid\lambda,\gamma) & \text{if } \Delta(D_0)=0, \\ p_{i\lambda}\theta + g_1(a_0\mid\lambda,\gamma) & \text{if } \Delta(D_0)>0, \\ \infty & \text{otherwise,} \end{cases} \qquad (12)$$

where $a_0 = a(i \mid \lambda, \gamma)$ and $D_0 = d_{a(i|\lambda,\gamma)|\lambda,\gamma}$.

4.2 Case 2: θ units of demand i are satisfied not only by period λ but also by some periods after λ

Suppose that the first replenishment after λ for demand i takes place in period $\tau \in [\lambda+1, \gamma-1]$. For notational convenience, we let $a_1 = a(i|\lambda, \tau)$ and $a_2 = a(i|\tau, \gamma)$. Based on these demands a_1 and a_2, we also let $D_1 = D(a_1|\lambda, \tau)$ and $D_2 = D(a_2|\tau, \gamma)$. Since the replenishments during $[\lambda+1, \gamma-1]$ are all FTL, Property 6 suggests that the total replenishment quantity is $x_{\tau,\gamma-1} = m(\theta + D_2)$. Let $\theta = \theta_1 + \theta_2$ where θ_1 denotes the amount of demand i replenished in period λ and θ_2 the amount replenished during $\{\tau, \tau+1, \ldots, \gamma-1\}$. Then, we can easily see that

$$\theta_1 = \theta - [m(\theta + D_2) - D_2] > 0, \text{ and}$$

$$\theta_2 = m(\theta + D_2) - D_2 > 0.$$

As was done in Subsection 4.1, here we also consider two subcases that period λ replenishes only demand i, and period λ replenishes not only demand i but also demands D_1. We consider the first case that period λ replenishes only demand i. In this case, we have $\Delta(D_1) = 0$ since demands D_1 all must be satisfied by FTL replenishments during periods $\{\lambda+1, \ldots, \tau\}$. Using similar arguments as in (12), we can see that

$$f(\theta, i|\lambda, \gamma) = K_\lambda + p_\lambda\theta_1 + g_0(a_1|\lambda, \tau) + f(\theta_2, i|\tau, \gamma).$$

Next consider the other case that period λ replenishes not only demand i but also demands D_1. In this case, we have $\Delta(D_1) > 0$ and

$$f(\theta, i|\lambda, \gamma) = p_\lambda\theta_1 + g_1(a_1|\lambda, \tau) + f(\theta_2, i|\tau, \gamma).$$

We use $f_2(\theta, i|\lambda, \gamma)$ to denote the cost of $f(\theta, i|\lambda, \gamma)$ in the case when the θ units of demand i are satisfied not only by period λ but also by some periods during $\{\lambda+1, \ldots, \gamma-1\}$. With the two formulas developed above, the recursion formula for $f_2(\theta, i|\lambda, \gamma)$ is given as

$$f_2(\theta,i\mid\lambda,\gamma) = \min_{\lambda<\tau\le L_i}\begin{cases} K_\lambda + p_\lambda\theta_1 + g_0(a_1\mid\lambda,\tau) + f(\theta_2,i\mid\tau,\gamma) & \text{if } \Delta(D_1)=0,\ \theta_1,\theta_2>0, \\ p_\lambda\theta_1 + g_1(a_1\mid\lambda,\tau) + f(\theta_2,i\mid\tau,\gamma) & \text{if } \Delta(D_1)>0,\ \theta_1,\theta_2>0, \\ \infty & \text{otherwise,} \end{cases} \qquad (13)$$

where $a_1 = a(i|\lambda, \tau)$, $a_2 = a(i|\tau, \gamma)$, D_1 is the total sum of demands $D(a_1|\lambda, \tau)$ and D_2 is the total sum of demands $D(a_2|\tau, \gamma)$. Furthermore, $\theta_1 = \theta - [m(\theta + D_2) - D_2]$ and $\theta_2 = m(\theta + D_2) - D_2$. Then, by (11), (12) and (13) the complete formula for $f(\theta, i|\lambda, \gamma)$ is given by

$$f(\theta,i\mid\lambda,\gamma) = \min\begin{cases} f_0(\theta,i\mid\lambda,\gamma), \\ f_1(\theta,i\mid\lambda,\gamma), \\ f_2(\theta,i\mid\lambda,\gamma). \end{cases}$$

Given a value θ and demand i, and periods λ and γ, the computation of $f(\theta, i|\lambda, \gamma)$ takes $O(T)$. If all the necessary costs $g_0(\cdot)$, $g_1(\cdot)$ have already been found, then the time for all $f(\theta, i|\lambda, \gamma)$ is $O(|\theta|nT^3)$, where $|\theta|$ denotes the needed value in computing an optimal solution, $0 < \theta \le d_i$. Let $S(i|\lambda, \gamma)$ be the set of *valid* values each θ can take for given demand i and periods λ and γ. Then, let's investigate what demands the set $S(i|\lambda, \gamma)$ contains. First of all, from (10), we know that each cost $f(d_i, i|\lambda, \gamma)$ should be computed. This implies $d_i \in S(i|\lambda, \gamma)$. In the computation of $f_1(\cdot)$ in (12), we note that no cost of $f(\cdot)$ is required. Consider formula (13) for $f_2(\cdot)$, which needs the cost $f(\theta_2, i|\tau, \gamma)$. Here, we have to note that $\theta_2 + D_2$ is a multiple of C, where D_2 is the total sum of demands $D(a_2|\tau, \gamma)$ and $a_2 = a(i|\tau, \gamma)$. Suppose that demand i has been supplied j times during $\{1, 2, \ldots, \tau-1\}$. Since the replenishments for demand i during $\{1, 2, \ldots, \tau-1\}$ are not major ones, each quantity distributed to demand i is less than the cargo capacity C (Property 7 and 8). In other words, the total sum of units of demand i dispatched during $\{1, 2, \ldots, \tau-1\}$ is less than jC. Let d_i' be the total amount of units of demand i dispatched during $\{1, 2, \ldots, \tau-1\}$. Then, we have $0 < d_i' < jC$. More precisely, we further suppose that $sC < d_i' < (s+1)C$ for some $s = 0, 1, \ldots, \tau-2$. We note that $m(d_i' + \theta_2 + D_2) = m(d_i') + \theta_2 + D_2$ since $\theta_2 + D_2$ is a multiple of C. That is, we have for some $s = 0, 1, \ldots, \tau-2$

$$m(d_i + D_2) = sC + \theta_2 + D_2.$$

From this equations, we can see that the valid set $S(i|\lambda, \gamma)$ also includes $m(d_i + D_2) - sC - D_2$ for some $s = 0, 1, \ldots, \tau-2$. Consequently, $S(i|\lambda, \gamma)$ is given as follows:

$$S(i|\lambda, \gamma) = \{d_i\}$$

$$\cup\ \{m(d_i + d_{a(i|\lambda,\gamma)|\lambda,\gamma}) - sC - d_{a(i|\lambda,\gamma)|\lambda,\gamma} : s = 0, 1, \ldots, \lambda-2\}.$$

Hence, the number of elements in the set $S(i|\lambda, \gamma)$ is at most $O(T)$. We therefore conclude that all the necessary computations for $f(\theta, i|\lambda, \gamma)$ require $O(nT^4)$.

6. Computing $g_0(i|\lambda, \gamma)$ and $g_1(i|\lambda, \gamma)$

In this section we describe how to compute $g_0(i|\lambda, \gamma)$ and $g_1(i|\lambda, \gamma)$.

Computing $g_0(i|\lambda, \gamma)$. Note that we have no demands $D(i|\lambda, \gamma)$ replenished in period λ. Supoose that the first replenishment period for the demand i occurs at period τ, $\lambda < \tau < \gamma$. Then demands $D(a(i|\lambda, \tau)|\lambda, \tau)$ are all replenished during $\{\lambda+1, \ldots, \tau-1\}$ and demands $D(i|\tau, \gamma)$ are all replenished during $\{\tau, \ldots, \gamma-1\}$. It is not hard to see that the costs for the periods $\{\lambda+1, \ldots, \tau-1\}$ and $\{\tau, \ldots, \gamma-1\}$ are $g_0(a(i|\lambda, \tau)|\lambda, \tau)$ and $f(d_i, i|\tau, \gamma)$, respectively. Hence, we have

$$g_0(i \mid \lambda,\gamma) = \min_{\lambda<\tau\le L_i} \{g_0(a(i \mid \lambda,\tau) \mid \lambda,\tau) + f(d_i,i \mid \tau,\gamma)\}$$

Computing $g_1(i|\lambda, \gamma)$. We first suppose that demand i is supplied by the replenishment in period λ. In this case, we have $g_1(i|\lambda, \gamma) = f(d_i, i|\lambda, \gamma)$. Next suppose that demand i is not replenished in period but replenished first in period τ, $\lambda < \tau < \gamma$. Then using similar arguments for $g_0(i|\lambda, \gamma)$, we have $g_1(i|\lambda, \gamma) = g_1(a(i|\lambda, \tau)|\lambda, \tau) + f(d_i, i|\tau, \gamma)$. Thus, we conclude that

$$g_1(i \mid \lambda,\gamma) = \min \begin{cases} f(d_i,i \mid \lambda,\gamma), \\ \min_{\lambda<\tau\le L_i} \{g_1(a(i \mid \lambda,\tau) \mid \lambda,\tau) + f(d_i,i \mid \tau,\gamma)\}. \end{cases}$$

We need to note that $g_0(i|\lambda, \gamma)$ and $g_1(i|\lambda, \gamma)$ are computed in $O(nT^3)$ if each $f(d_i, i|\tau, \gamma)$ is preprocessed. Furthermore, the computations of $g_0(i|\lambda, \gamma)$ and $g_1(i|\lambda, \gamma)$ require $f(\theta, i|\tau, \gamma)$ where $\theta = d_i$. Hence, such θs belong to the valid set of $S(i|\tau, \gamma)$. Consequently, an optimal algorithm is found in $O(nT^4)$.

7. Conclusion

In this paper we provided an optimal $O(nT^4)$ algorithm for a dynamic lot-sizing model for inbound shipment planning with production time window demands in a third-party warehouse hub. The algorithm much relys on the special nonspeculative cost structure. The future study needs to explore what algorithm is possible for more general cost structures.

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