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Subsethood Measure and Entropy of Interval-valued Intuitionistic Fuzzy Sets

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1 Introduction

As a generalization of an interval-valued fuzzy set (IVFS), Atanassov and Gargov [2] introduced the notion of interval-valued intuitionstic fuzzy set (IVIFS), which is characterized by a membership function and a non-membership function whose values are intervals rather then exact numbers. Some authors have investigated IVIFSs and its relevant topics. Atanassov [1] gave some relations and operations over IVIFSs, and studied their basic properties. Bustince and Burillo [4] defined the concepts of correlation and correlation coefficient of IVIFSs, and obtained two decomposition theorems of the correlation of IVIFSs in terms of the correlation of IVIFSs, the entropy of intuitionistic fuzzy sets (A-IFSs) and the correlation of A-IFSs. The results of Bustince and Burillo's in a general probability space are extended by Hong [6]. Hung and Wu [7] and Xu [15] proposed other new approaches to deriving the correlation coefficients of IVIFSs.

Recently, Park et al. [11] extended three methods for measuring distances between IVIFSs to IVIFSs and showed that these reduce to the Burillo and Bustince's distances [3], proposed distances and Grzegorzewski's distances [5], respectively, for IVIFSs. Furthermore, Park et al. [12] studied the relationship between entropy and similarity measure of IVIFSs, gave three theorems that entropy and similarity measure of IVIFSs can be transformed by each other based on their axiomatic definitions and proposed some formulas to calculate entropy and similarity measure of IVIFSs.

In this paper, we establish a unified framework between the concepts of subsethood, entropy and cardinality for IVIFSs. Then we review the axioms of subsethood for IVIFSs and propose an alternative axiomatic skeleton, in order for subsethood to reduce to entropy. Based on the axioms, we also prove an interval-valued intuitionistic version of the entropy-subsethood theorem and derive new measures of subsethood and entropy for IVIFSs. Furthermore, the concepts of cardinality and average possible cardinality of IVIFSs is presented. We carry out an algebraic and geometrical analysis, which demonstrates a connection between the above-mentioned cardinality and the least and biggest cardinalities. Finally, based on the average possible cardinality, we extend the fuzzy entropy theorem in the interval-valued intuitionistic fuzzy setting and provide connections between the proposed measure and corresponding measures for IVIFSs and fuzzy sets.

2 Basic notions

Throughout this paper, X denotes the universe set, IVIFS(X), IVFS(X), A-IFS(X) and FS(X) stand for the set of all IVIFSs, IVFSs, A-IFSs and fuzzy sets on X, respectively. The operation `` c'' is the complement of IVIFS, IVFS, A-IFS or fuzzy set on X.

Let I=[0,1] and $[I]$ be the set of all closed subintervals of the interval $[0,1]$. Then, by Zadeh's extension principle [16], we can popularize these operations such as $\vee, \wedge$ and c to $[I]$ and thus $([I],\vee,\wedge,c)$ is a complete lattice with a minimal element $\overline{0}=[0,0]$ and a maximal element $\overline{1}=[1,1]$. Furthermore, let $\overline{a}=[a^-, a^+]$, $\overline{b}=[b^-, b^+]$, then we have $\overline{a}=\overline{b} \Leftrightarrow a^-=b^-$, $a^+=b^+$, $\overline{a}\le\overline{b} \Leftrightarrow a^-\le b^-$, $a^+\le b^+$, and $\overline{a}<\overline{b} \Leftrightarrow \overline{a}\le\overline{b}$ and $\overline{a}\neq\overline{b}$.

Definition 1. An interval-valued intuitionisti-c fuzzy set A on X is defined as

$$A=\{(x, M_A(x),\ N_A(x)): x\in X\}, \qquad (1)$$

where $M_A: X\to[I]$ and $N_A: X\to[I]$ denote, respectively, membership function and non-membership function of A and satisfy $0\le M_A^+(x)+N_A^+(x)\le 1$ for any $x\in X$.

By $\Upsilon=\{(x,[1,1],[0,0]): x\in X\}$ and $O=\{(x,[0,0],[1,1]): x\in X\}$, we denote the greatest and the smallest IVIFSs, respectively, and denote frequently $\Upsilon=([1,1],[0,0])$ and $O=([0,0],[1,1])$ for simplicity. For $A\in$ IVIFS(X), let $A=([M_A^-, M_A^+],[N_A^-, N_A^+])$. Then $A_L=\langle M_A^-, N_A^-\rangle$ is called lower A-IFS of A and $A_U=\langle M_A^+, N_A^+\rangle$ is called upper A-IFS of A. Thus, under these notions, we can give another representation of an IVIFS A as $A=(A_L, A_U)$. This representation gives us to describe the pseudo average possible cardinality of an IVIFS in the Euclidean plane, as it will be demonstrated in Section 3.

3 Cardinality for IVIF sets

Definition 2. For a set $A\in$ IVIFS(X) the following two cardinalities are defined:

- the least cardinality or min-sigma-count, which is given by

$$\min\sum Count(A)=\sum_{x_i\in X}\frac{M_A^+(x_i)+M_A^-(x_i)}{2} \qquad (2)$$

- the biggest cardinality or max-sigma-count defined as

$$\max\sum Count(A)=\sum_{x_i\in X}\frac{2-(N_A^+(x_i)+N_A^-(x_i))}{2}. \qquad (3)$$

The cardinality of the IVIFS A is defined as the interval

$$card(A)=[\min\sum Count(A), \max\sum Count(A)]. \qquad (4)$$

For the smallest IVIFS O, equivalenty to Vlachos and Sergiadis's definition of cardinality for A-IFSs, we call the magnitude $M(A)$ of the vector $\overrightarrow{OA}$, using Hamming distance, the *average possible cardinality* of the IVIFS A. The characterization of *average possible cardinality* will be justified by the following analysis. The Hamming distance between two IVIFSs A and B is given [11]

$$d_1'(A,B)=\frac{1}{4}\sum_{i=1}^{n}(|M_A^-(x_i)-M_B^-(x_i)|+|M_A^+(x_i)-M_B^+(x_i)|+|N_A^-(x_i)-N_B^-(x_i)|+|N_A^+(x_i)-N_B^+(x_i)|) \qquad (5)$$

Definition 3. For a set $A\in$ IVIFS(X) the average possible cardinality $M(A)$ is defined as

$$M(A)=d_1'(O,A)=\frac{1}{4}\sum_{x_i\in X}(M_A^-(x_i)+1-N_A^-(x_i)+M_A^-(x_i)+1-N_A^-(x_i)). \qquad (6)$$

From (6), taking into account (2), it follows that $M(A)$ is the midpoint of the interval $[\min\sum Count(A), \max\sum Count(A)]$. So, (6) encompasses the notions of least, biggest and average possible cardinalities. Connections between the aforementioned cardinalities will be discussed and applied in interval-valued intuitionistic fuzzy decision making problems.

The axiomatic definition of cardinality and average possible cardinality of IVIFSs can be extended from A−IFS theory or IVFS theory. Specially, if an IVIFS A becomes an A−IFS, then M(A) is average possible cardinality of A−IFS. To derive connection between the notions of cardinality for IVIFSs and A−IFS, for $A \in$ IVIFS(X), we consider the lower A−IFS $A_L = \langle M_A^-, N_A^- \rangle$ of A and upper A−IFS $A_U = \langle M_A^+, N_A^+ \rangle$ of A. From (6) and Definition 18 of [18], we obtain

$$M(A) = \frac{1}{2} \sum_{x_i \in X} \left(\frac{M_A^-(x_i) + M_A^+(x_i)}{2} + \frac{2 - N_A^-(x_i) - N_A^+(x_i)}{2} \right) = \frac{1}{2} \sum_{x_i \in X} \left(M_A^-(x_i) + \frac{\pi_{A_L}(x_i)}{2} + M_A^+(x_i) + \frac{\pi_{A_U}(x_i)}{2} \right) = \frac{M(A_L) + M(A_U)}{2}. \quad (7)$$

Thus, the average possible cardinality of IVIFS A is half of the sum of the average possible cardinalities of A_L and A_u.

4 Subsethood for IVIFSs

Definition 4. A real function S: IVIFS(X) × IVIFS(X) → [0,1] is called a subsethood measure of IVIF sets if S satisfies the following properties:

(S1) $S(A,B) = 1$ iff $A \subset B$;
(S2) If $A^c \subset A$, then $S(A, A^c) = 0$ iff $A = \Upsilon$;
(S3) If $B \subset A_1 \subset A_2 \Rightarrow S(A_1, B) \geq S(A_2, B)$ and if $B_1 \subset B_2$, then $S(A, B_1) \leq S(A, B_2)$.

Theorem 1. *For two IVIFSs A and B on finite universe X,*

$$S_{\text{IVIFS}}(A,B) = 1 - \Big[\sum_{x_i \in X} \big(\max\{0, M_A^-(x_i) - M_B^-(x_i)\} + \max\{0, M_A^+(x_i) - M_B^+(x_i)\} + \max\{0, N_A^-(x_i) - N_A^-(x_i)\} + \max\{0, N_B^+(x_i) - N_A^+(x_i)\}\big)\Big] \div \Big[\sum_{x_i \in X} (2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i))) \Big] \quad (8)$$

is a subsethood measure of IVIFS*s.*

Remark 1. Note that if $A = O$, (8) is undefined, due to the fact that M(O) = 0. So, by definition, for any IVIFS B, we have that $S_{\text{IVIFS}}(O, B) = 1$, since O is a proper subset of any IVIFS.

5 Entropy of IVIFSs

Definition 5. A real function E: IVIFS(X) → [0,1] is called an entropy on IVIFS(X) if E satisfies the following properties:

(E1) $E(A) = 0$ iff A is a crisp set ;
(E2) $E(A) = 1$ iff $M_A(x) = N_A(x)$ for all $x \in X$;
(E3) $E(A) \leq E(B)$ if A refines B ;
(E4) $E(A) = E(A^c)$.

Generalizing the works of Kosko [8−10] and Vlachos and Sergiadis [14], we state the entropy−subsethood theorem for IVIFS*s*, based on the axiomatic skeleton (S1)−(S3).

Theorem 2. *Suppose S is a subsethood measure of* IVIFS*s on X and $A \in$* IVIFS *(X), then*

$$E(A) = S(A \cup A^c, A \cap A^c) \quad (9)$$

is an entropy measure of A.

Now, we state a relationship between the entropy and average possible cardinality of IVIFS*s*, which generalize the works of Kosko [8−10] and Vlachos and Sergiadis [14].

Theorem 3. *Suppose* M *is an average possible cardinality of* IVIFS*s on X and $A \in$* IVIFS *(X), then*

$$E(A) = \frac{M(A \cap A^c)}{M(A \cup A^c)} \quad (10)$$

is an entropy measure of A.

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