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Entropy and Similarity Measure of Interval-valued Intuitionistic Fuzzy Sets

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Abstract

In this paper, we introduce concepts of entropy and similarity measure of interval-valued intuitionistic fuzzy sets (IVIFSs), discuss their relationship between similarity measure and entropy of IVIFSs, show that similarity measure and entropy of IVIFSs can be transformed by each other based on their axiomatic definitions and give some formulas to calculate entropy and similarity measure of IVIFSs.

1 Introduction

Some authors have investigated interval-valued fuzzy sets (IVFSs) and/or intuitionistic fuzzy sets (IFSs) and its relevant topics. For example, Burillo and Bustince [1] and Szmidt and Kacprzsk [7] researched entropy of IVFSs and/or IFSs from different point of views, respectively. Li and Cheng [3] proposed similarity measures of IFSs and applied these measures to pattern recognition. Liang and Shi [4], Mitchell [5] and Park et al [6] pointed out that Li and Cheng's measures are not always effective in some cases and made some modifications, respectively. In 2004, Grzegorzewski [2] studied distance between IVFSs based on the Hausdorff metric, Zeng and Li [9] studied the relationship between entropy and similarity measure of IVFSs, and Vlachos and Sergiadis [8] proposed the definition of subsethood of IVFSs and discussed its relationship with entropy and cardinality.

In this paper, we want to study the relationship between entropy and similarity measure of IVIFSs, give three theorems that entropy and similarity measure of IVIFSs can be transformed by each other based on their axiomatic definitions and propose some formulas to calculate entropy and similarity measure of IVIFSs.

2 Basic notions

Definition 1. An interval-valued intuitionistic fuzzy set (IVIFS, for short) A on X is defined as

$$A=\{(x,\ M_A(x),\ N_A(x)) : x \in X\},$$

where $M_A : X \to [I]$ and $N_A : X \to [I]$ denote, respectively, membership function and non-membership function of A and satisfy $0 \le M_A^+(x)+N_A^+(x) \le 1$ for any $x \in X$. For simplicity, we often denote $A=(M_A, N_A)$.

The basic operations such as union, intersection and complement are defined as follows: let $A,\ B \in \mathrm{IVIFS}(X)$, then

- $A \cup B=\{(x,[\max(M_A^-(x), M_B^-(x)),$ $\max(M_A^+(x), M_B^+B(x))],\ \min(N_A^-(x), N_B^-(x)),$ $\min(N_A^+(x), N_B^+B(x))]) : x \in X\}$,
- $A \cap B=\{(x,[\min(M_A^-(x), M_B^-(x)),$ $\min(M_A^+(x), M_B^+B(x))],\ \max(N_A^-(x), N_B^-(x)),$ $\max(N_A^+(x), N_B^+B(x))]) : x \in X\}$,
- $A \subset B \ \leftrightarrow\ M_A(x) \le M_B(x),\ N_A(x) \ge M_B(x),$ for all $x \in X$,
- $A=B \ \leftrightarrow\ A \subset B,\ B \subset A$,
- $A^c = \{(x,\ N_A(x),\ M_A(x)) : x \in X\}$.

Definition 2. Let $A,\ B \in \mathrm{IVIFS}(X)$. We call that A refines B (i.e., A is less fuzzy than B), denoted as $A \le B$, if the following conditions are satisfied: for every $x \in X$,
(a) If $M_B(x) \ge N_B(x)$, then $M_A(x) \ge M_B(x)$

and $N_A(x) \le N_B(x)$;
(b) If $M_B(x) \le N_B(x)$, then $M_A(x) \le M_B(x)$ and $N_A(x) \ge N_B(x)$.

Theorem 1. *If A refines B, $A, B \in IVIFS(X)$, then we have* $A \cap A^c \subset B \cap B^c$ *and* $A \cup A^c \supset B \cup B^c$.

3 Entropy of IVIFSs

Definition 3. A real function E : IVIFS(X) $\to [0,1]$ is called entropy on IVIFS(X) if E satisfies the following properties:
(E1) $E(A)=0 \leftrightarrow A$ is a crisp set;
(E2) $E(A)=1 \leftrightarrow M_A(x)=N_A(x)$;
(E3) $E(A) \le E(B)$ if A refines B ;
(E4) $E(A)=E(A^c)$.

Then we can give the following formulas to calculate entropy of IVIFS A :

$$E_1(A)=1-\frac{1}{2n}\sum_{i=1}^{n}\left(|M_A^-(x_i)-N_A^-(x_i)| + |M_A^+(x_i)-N_A^+(x_i)|\right), \quad (1)$$

$$E_2(A)=1-\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left((M_A^-(x_i)-N_A^-(x_i))^2 + (M_A^+(x_i)-N_A^+(x_i))^2\right)}, \quad (2)$$

$$E_3(A)=1-\frac{1}{2(b-a)}\int_a^b \left(|M_A^-(x)-N_A^-(x)| + |M_A^+(x)-N_A^+(x)|\right)dx, \quad (3)$$

$$E_4(A)=\left\{\int_a^b\left(\min(M_A^-(x), N_A^-(x)) + \min(M_A^+(x), N_A^+(x))\right)dx\right\} \div \left\{\int_a^b\left(\max(M_A^-(x), N_A^-(x)) + \max(M_A^+(x), N_A^+(x))\right)dx\right\} \quad (4)$$

where the integral in Equations (3) and (4) is Lebesque integral.

Definition 4. A real function S: IVIFS$(X) \times$ IVIFS$(X) \to [0,1]$ is called similarity measure of IVIFSs if S satisfies the following properties:
(S1) $S(A, A^c)=0$ if A is a crisp set;
(S2) $S(A,B)=1 \leftrightarrow A=B$;
(S3) $S(A,B)=S(B,A)$;
(S4) for all $A,B,C \in$ IVIFS(X), if $A \subset B \subset C$, then $S(A,C) \le S(A,B)$, $S(A,C) \le S(B,C)$.

Then we can give the following formulas to calculate similarity measure of IVIFSs A and B :

$$S_1(A,B)=1-\frac{1}{4n}\sum_{i=1}^{n}\left(|M_A^-(x_i)-M_B^-(x_i)| + |M_A^+(x_i)-M_B^+(x_i)| + |N_A^-(x_i)-N_B^-(x_i)| + |N_A^+(x_i)-N_B^+(x_i)|\right) \quad (5)$$

$$S_2(A,B)=1-\left\{\frac{1}{4n}\sum_{i=1}^{n}\left((M_A^-(x_i)-M_B^-(x_i))^2 + (M_A^+(x_i)-M_B^+(x_i))^2 + (N_A^-(x_i)-N_B^-(x_i))^2 + (N_A^+(x_i)-N_B^+(x_i))^2\right)\right\}^{\frac{1}{2}} . \quad (6)$$

Obviously, the axiomatic definition of similarity measure and entropy of IVIFSs can be extended from IFS theory or IVFS theory. Specially, if IVIFSs A and B become IFSs (resp. IVFSs), then $S(A,B)$ is similarity measure of IFSs (resp. IVFSs). Based on this point of view, we have the following results.

Theorem 2. If $A \in$ IVIFS(X), then

$$E(A)=\frac{S(A_-,(A_-)^c)+S(A_+,(A_+)^c)}{2}$$

is entropy of IVIFS A, where A_- and A_+ are IFSs given by $A_-=(M_A^-, N_A^-)$ and $A_+=(M_A^+, N_A^+)$, respectively.

Theorem 3. If $A \in$ IVIFS(X) satisfies that IVFS N_A is the complement of IVFS M_A, then $S(M_A,N_A)$ is entropy of IVIFS A.

4 Relationship between similarity measure and entropy of IVIFSs

Considering that the real functions of similarity measure and entropy of IVIFSs are not unique, in this section, we will discuss the relationship between similarity measure and entropy of IVIFSs based on their axiomatic definitions.

First, we propose a transform method of setting up similarity measure of IVIFSs based on entropy of IVIFS. For $A,B \in$ IVIFS(X), we define an IVIFS $I(A,B)$

on X as follows: for every $x \in X$,

$$M^-_{I(A,B)}(x) = \frac{1}{2}[1 + \min\{\min(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \min(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}],$$

$$M^+_{I(A,B)}(x) = \frac{1}{2}[1 + \max\{\min(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \min(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}],$$

$$N^-_{I(A,B)}(x) = \frac{1}{2}[1 - \max\{\max(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \max(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}],$$

$$N^+_{I(A,B)}(x) = \frac{1}{2}[1 - \max\{\min(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \min(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}].$$

Then we have the following result.

Theorem 4. *Suppose E be an entropy of* IVIFSs *on X, for $A, B \in$ IVIFS(X), then $E(I(A,B))$ is similarity measure of* IVIFSs *A and B.*

Corollary 1. *Suppose E be entropy of* IVIFSs *on X, $I(A,B)$ is defined as above, then $E((I(A,B)))^c$ is similarity measure of* IVIFSs *A and B.*

For $A, B \in$ IVIFS(X), we define an IVIFS $J(A,B)$ on X as follows: for every $x \in X$ and $p > 0$,

$$M^-_{I(A,B)}(x) = \frac{1}{2}[1 + \min\{\min(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \min(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}],$$

$$M^+_{I(A,B)}(x) = \frac{1}{2}[1 + \max\{\min(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \min(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}],$$

$$N^-_{I(A,B)}(x) = \frac{1}{2}[1 - \max\{\max(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \max(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}],$$

$$N^+_{I(A,B)}(x) = \frac{1}{2}[1 - \max\{\min(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \min(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}].$$

Then we have

Corollary 2. *Suppose E be entropy of* IVIFSs *on X, $J(A,B)$ is defined as above, then both $E(J(B,B))$ and $((J(A,B))^c)$ are similarity measures of* IVIFSs *A and B.*

Example 1. Let $X = \{x_1, x_2, \cdots, x_n\}$, $A, B \in$ IVIFS(X) and

$$E(A) = 1 - \frac{1}{2n}\sum_{i=1}^{n}(|M^-_A(x_i) - N^-_A(x_i)| + |M^+_A(x_i) - N^+_A(x_i)|).$$

Then

$$E(I(A,B)) = 1 - \frac{1}{4n}\sum_{i=1}^{n}(\min(|M^-_A(x_i) - M^-_B(x_i)|, |M^+_A(x_i) - M^+_B(x_i)|, |N^-_A(x_i) - N^-_B(x_i)|, |N^+_A(x_i) - N^+_B(x_i)|) + \max(|M^-_A(x_i) - M^-_B(x_i)|, |M^+_A(x_i) - M^+_B(x_i)|, |N^-_A(x_i) - N^-_B(x_i)|, |N^+_A(x_i) - N^+_B(x_i)|) + 2\max\{\min(|M^-_A(x_i) - M^-_B(x_i)|, |M^+_A(x_i) - M^+_B(x_i)|), \min(|N^-_A(x_i) - N^-_B(x_i)|, |N^+_A(x_i) - N^+_B(x_i)|)\})$$

is similarity measure of IVIFSs A and B.

Next, we propose another method of setting up entropy of IVIFS based on similarity measure of IVIFSs. For $A \in$ IVIFS(X), we define IVIFSs $m(A)$ and $n(A)$ on X as follows: for every $x \in X$,

$$M^-_{m(A)}(x) = \frac{1 + (M^-_A(x) - N^-_A(x))^4}{2},$$

$$M^+_{m(A)}(x) = \frac{1 + (M^-_A(x) - N^-_A(x))^2}{2},$$

$$N^-_{m(A)}(x) = \frac{1 - |M^-_A(x) - N^-_A(x)|}{2},$$

$$N^+_{m(A)}(x) = \frac{1 - (M^-_A(x) - N^-_A(x))^2}{2},$$

$$M^-_{n(A)}(x) = \frac{1 - |M^+_A(x) - N^+_A(x)|}{2},$$

$$M^+_{n(A)}(x) = \frac{1 - (M^+_A(x) - N^+_A(x))^2}{2},$$

$$N^-_{n(A)}(x) = \frac{1 + (M^+_A(x) - N^+_A(x))^4}{2},$$

$$N^+_{n(A)}(x) = \frac{1 + (M^+_A(x) - N^+_A(x))^2}{2}.$$

Then we have the following result.

Theorem 5. *Suppose* S *be similarity measure of* IVIFSs *on* X, $A \in \text{IVIFS}(X)$, *then* $S(m(A), n(A))$ *is entropy of* IVIFS A.

Corollary 3. *Suppose* S *be similarity measure of* IVIFSs *on* X, $m(A)$ *and* $n(A)$ *are defied as above, then* $S((m(A))^c, (n(A))^c)$ *is entropy of* IVIFS A.

Example 2. Let $X = \{x_1, x_2, \cdots, x_n\}$, $A \in IVIFS(X)$ and

$$S(A,B) = 1 - \frac{1}{4n}\sum_{i=1}^{n} (| M_A^-(x_i) - M_B^-(x_i) | + | M_A^+(x_i) - M_B^+(x_i) | + | N_A^-(x_i) - N_B^-(x_i) | + | N_A^+(x_i) - N_B^+(x_i) |).$$

Then

$$S(m(A), n(A)) = 1 - \frac{1}{8n}\sum_{i=1}^{n} ((M_A^-(x_i) - N_A^-(x_i))^4 + 2(M_A^-(x_i) - N_A^-(x_i))^2 + | M_A^-(x_i) - M_B^-(x_i) | + (M_A^+(x_i) - N_A^+(x_i))^4 + 2(M_A^+(x_i) - N_A^+(x_i))^2 + | M_A^+(x_i) - M_B^+(x_i) |)$$

is entropy of IVIFS A.

Theorem 6. *Suppose* S *be similarity measure of* IVIFSs *on* X, $A \in \text{IVIFS}(X)$, *then* $S(A, A^c)$ *is entropy of* IVIFS A.

Example 3. Let $X = \{x_1, x_2, \cdots, x_n\}$, $A, B \in$ *IVIFS*(X) and

$$S(A,B) = 1 - \frac{1}{4n}\sum_{i=1}^{n} (| M_A^-(x_i) - M_B^-(x_i) | + | M_A^+(x_i) - M_B^+(x_i) | + | N_A^-(x_i) - N_B^-(x_i) | + | N_A^+(x_i) - N_B^+(x_i) |).$$

Then

$$S(A, A^c) = 1 - \frac{1}{2n}\sum_{i=1}^{n} (| M_A^-(x_i) - N_A^-(x_i) | + | M_A^+(x_i) - N_A^+(x_i) |) = E_1(A)$$

is entropy of A.

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