Controllability for the Semilinear Fuzzy Integrodifferential Equations with Nonlocal Conditions and Forcing Term with Memory

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Abstract

In this paper. we study the controllability for the semilinear fuzzy integrodifferential equations with nonlocal condition and forcing term with memory in E_N by using the concept of fuzzy number whose values are normal, convex, upper semicontinuous and compactly supported interval in E_N.

Key Words : fuzzy number, semilinear, integrodifferential equation, nonlocal.

1. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva [3] studied the existence and uniqueness of solution for the fuzzy differential equation on E_N where E_N is normal, convex, upper semicontinuous and compactly supported fuzzy sets in R_N. Seikkala [8] proved the existence and uniqueness of fuzzy solution for the following equation:

$$\dot{x}(t) = f(t,x(t)), \quad x(0) = x_0,$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number in E^1. Diamond and Kloeden [2] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E_N^1 using by Kuhn-Tucker theorems. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently, Kwun etal. [5] proved the existence and uniqueness of solutions for the following semilinear fuzzy integrodifferential equations with nonlocal initial conditions and forcing term with memory $(u(t) \equiv 0)$:

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds \right] \tag{1}$$

$$+ f(t, x, \int_0^t k(t, s, x(s))ds) + u(t), \; t \in I = [0, T],$$

$$x(0) + g(x) = x_0 \in E_N , \tag{2}$$

where $A : I \rightarrow E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α-level intervals, $f : I \times E_N \times E_N \rightarrow E_N$ and $k : I \times I \times E_N \rightarrow E_N$ are nonlinear continuous functions, $G(t)$ is $n \times n$ continuous matrix such that $\frac{dG(t)x}{dt}$ is continuous for $x \in E_N$ and $t \in I$ with $\| G(t) \| \leq k$, $k > 0$, $u : I \rightarrow E_N$ is control function and $g : E_N \rightarrow E_N$ is a nonlinear continuous function.

In this paper, we study the controllability for the above semilinear fuzzy integrodifferential equations with nonlocal condition and forcing term with memory (1)-(2) in E_N .

2. Preliminaries

We denote the suprimum metric d_∞ on E_N and the suprimum metric H_1 on $C(I : E^n)$.

Definition 2.1. Let a, $b \in E^n$.

$d_\infty(a, b) = \sup\{d_H([a]^\alpha, [b]^\alpha) : \alpha \in (0, 1]\}$

where d_H is the Hausdorff distance.

Definition 2.2. Let $x, y \in C(I : E^n)$.

$$H_1(x, y) = \sup\{d_\infty(x(t), y(t)) : t \in I\}.$$

Let I be a real interval. A mapping $x : I \to E_N$ is called a fuzzy process. We denote

$$[x(t)]^\alpha = [x_l^\alpha(t), x_r^\alpha(t)],\ t \in I,\ 0 < \alpha \le 1.$$

The derivative $x'(t)$ of a fuzzy process x is defined by

$$[x'(t)]^\alpha = [(x_l^\alpha)'(t), (x_r^\alpha)'(t)],\ 0 < \alpha \le 1$$

provided that is equation defines a fuzzy $x'(t) \in E_N$.

The fuzzy integral

$$\int_a^b x(t)dt,\quad a, b \in I$$

is defined by

$$\left[\int_a^b x(t)dt\right]^\alpha = \left[\int_a^b x_l^\alpha(t)dt,\ \int_a^b x_r^\alpha(t)dt\right]$$

provided that the Lebesgue integrals on the right exist.

Definition 2.3. [1] The fuzzy process $x : I \to E_N$ is a solution of equations (1)-(2) without the inhomogeneous term if and only if

$$(\dot{x}_l^\alpha)(t) = \min\Big\{A_l^\alpha(t)[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds],\ i, j = l, r\Big\},$$

$$(\dot{x}_r^\alpha)(t) = \max\Big\{A_r^\alpha(t)[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds],\ i, j = l, r\Big\},$$

and

$$(x_l^\alpha)(0) = x_{0l}^\alpha - g_l^\alpha(x),$$

$$(x_r^\alpha)(0) = x_{0r}^\alpha - g_r^\alpha(x),$$

Now we assume the following:

(H1) The nonlinear function $f : [0, T] \times E_N \times E_N \to E_N$ satisfies a global Lipschitz condition, there exists a finite constants k_1, $k_2 > 0$ such that

$$d_H([f(s, \xi_1(s), \eta_1(s))]^\alpha, [f(s, \xi_2(s), \eta_2(s))]^\alpha) \le k_1 d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) + k_2 d_H([\eta_1(s)]^\alpha, [\eta_2(s)]^\alpha)$$

for all $\xi_1(s)$, $\xi_2(s)$, $\eta_1(s)$, $\eta_2(s) \in E_N$.

(H2) The nonlinear function $k : [0, T] \times [0, T] \times E_N \to E_N$ satisfies a global Lipschitz condition, there exists a finite constant $M > 0$ such that

$$d_H([k(t, s, \psi_1(s))]^\alpha, [k(t, s, \psi_2(s))]^\alpha) \le M d_H([\psi_1(s)]^\alpha, [\psi_2(s)]^\alpha)$$

for all $\psi_1(s), \psi_2(s) \in E_N$.

(H3) The nonlinear function $g : E_N \to E_N$ satisfies following inequality

$$d_H([g(\xi_1)]^\alpha, [g(\xi_2)]^\alpha) \le L\, d_H([\xi_1(\cdot)]^\alpha, [\xi_2(\cdot)]^\alpha),$$

where constant $L > 0$.

(H4) $S(t)$ is a fuzzy number satisfying, for $y \in E_N$ and $s'(t)y \in C^1(I : E_N) \cap C(I : E_N)$, the equation

$$\frac{d}{dt}S(t)y = A\left[S(t)y + \int_0^t G(t-s)S(s)yds\right] = S(t)Ay + \int_0^t S(t-s)AG(s)yds,\ t \in I,$$

such that

$$[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)],$$

and $S_i^\alpha(t)$ $(i = l, r)$ is continuous. That is, there exists a constant $c > 0$ such that $|S_i^\alpha(t)| \le c$ for all $t \in I$.

(H5) $c(L + k_1 T + k_2 M T^2) < 1$.

3. Nonlocal controllability

In this section, we consider the controllability for the equations (1)-(2).

The equations (1)-(2) is related to the following fuzzy integral equation:

$$x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s)u(s)ds + \int_0^t S(t-s)f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)ds, \quad (3)$$

where S(t) is satisfy (H4).

Theorem 3.1. [5]. Let $T>0$, assume that the function f, k and g satisfy hypotheses (H1)-(H5). Then, for every $x_0 \in E_N$, equation $(3)(u(t)\equiv 0)$ has a unique fuzzy solution $x\in C([0,T]:E_N)$.

Definition 3.2. The equation (3) is nonlocal controllable if, there exists $u(t)$ such that the fuzzy solution $x(t)$ of (3) satisfies $x(T)=x^1-g(x)$ (i.e., $[x(T)]^\alpha=[x^1-g(x)]^\alpha$) where x^1 is target set.

Defined the fuzzy mapping $G: \tilde{P}(R)\to E_N$ by

$$G^\alpha(v)=\begin{cases}\int_0^T S^\alpha(T-s)v(s)ds\,, & v\subset \overline{\Gamma_u},\\ 0, & \text{otherwise}.\end{cases} \qquad (4)$$

Then there exists G_i^α $(i=l,r)$ such that

$$G_l^\alpha(v_l)=\int_0^T S_l^\alpha(T-s)v_l(s)ds, v_l(s)\in[u_l^\alpha(s),u^1(s)],$$

$$G_r^\alpha(v_r)=\int_0^T S_r^\alpha(T-s)v_r(s)ds, v_r(s)\in[u^1(s),u_r^\alpha(s)].$$

We assume that G_l^α, G_r^α are bijective mappings.

Hence α-level of $u(s)$ are

$$\begin{aligned}[u(s)]^\alpha &= [u_l^\alpha(s), u_r^\alpha(s)]\\ &= [(\widetilde{G_l^\alpha})^{-1}((x^1)_l^\alpha-g_l^\alpha(x)-S_l^\alpha(T)(x_{0l}^\alpha-g_l^\alpha(x))),\\ &\quad (\widetilde{G_r^\alpha})^{-1}((x^1)_r^\alpha-g_r^\alpha(x)-S_r^\alpha(T)(x_{0r}^\alpha-g_r^\alpha(x)))].\end{aligned}$$

Thus we can be introduced $u(s)$ of nonlinear system

$$\begin{aligned}[u(s)]^\alpha &= [u_l^\alpha(s), u_r^\alpha(s)]\\ &= \Big[(\widetilde{G_l^\alpha})^{-1}((x^1)_l^\alpha-g_l^\alpha(x)-S_l^\alpha(T)\\ &\quad\times(x_{0l}^\alpha-g_l^\alpha(x))-\int_0^T S_l^\alpha(T-s)\\ &\quad\times f_l^\alpha(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)ds)),\\ &\quad(\widetilde{G_r^\alpha})^{-1}((x^1)_r^\alpha-g_r^\alpha(x)-S_r^\alpha(T)\\ &\quad\times(x_{0r}^\alpha-g_r^\alpha(x))-\int_0^T S_r^\alpha(T-s)\\ &\quad\times f_r^\alpha(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)ds)\Big].\end{aligned}$$

Notice that $\Phi_x(T)=x^1-g(x)$, which means that the control $u(t)$ steers the equation (3) from the origin to $x^1-g(x)$ in time T provided we can obtain a fixed point of the nonlinear operator Φ.

Assume that the following hypotheses:

(H6) Linear system of equation (3) $(f=0)$ is nonlocal controllable.

(H7) $cL(1+(1+c)T)+(k_1+k_2MT)T(1+cT)<1$.

Theorem 3.3. Suppose that hypotheses (H1)-(H7) are satisfied. Then the equation (3) is nonlocal controllable.

4. Example

Consider the semilinear one dimensional heat equation on a connected domain $(0,1)$ for a material with memory, boundary condition $x(t,0)=x(t,1)=0$ and with initial condition $x(0,z)=x_0(z)$, $\sum_{k=1}^p c_k x(t_k,z)=g(x)$, where $x_0(z)\in E_N$. Let $x(t,z)$ be the internal energy and $f(t,x(t,z),\int_0^t k(t,s,x(t,z))ds)=\tilde{2}tx(t,z)^2+\int_0^t(t-s)x(s)ds$ be the external heat with memory.

Let $A=\tilde{2}\frac{\partial^2}{\partial z^2}$ and $G(t-s)=e^{-(t-s)}$, then the balance equation becomes

$$\frac{dx(t)}{dt}=\tilde{2}[x(t)-\int_0^t e^{-(t-s)}x(s)ds] \qquad (5)$$
$$+\tilde{2}tx(t)^2+\int_0^t(t-s)x(s)ds+u(t),\ t\in I,$$

$$x(0)=x_0-\sum_{k=1}^p c_k x(t_k,z). \qquad (6)$$

Since α-level set of fuzzy number $\tilde{2}$ is $[2]^\alpha=[\alpha+1,3-\alpha]$ for all $\alpha\in[0,1]$, α-level set of $f(t,x(t),\int_0^t k(t,s,x(s))ds)$ is

$$[f(t,x(t),\int_0^t k(t,s,x(s))ds)]^\alpha$$

$$= [t(\alpha+1)(x_l^\alpha(t))^2 + \int_0^t (t-s)x_l^\alpha(t),$$
$$t(3-\alpha)(x_r^\alpha(t))^2 + \int_0^t (t-s)x_r^\alpha(t)].$$

Further, we have

$$d_H([f(t, x(t), \int_0^t k(t, s, x(s))ds)]^\alpha,$$
$$[f(t, y(t), \int_0^t k(t, s, y(s))ds)]^\alpha)$$
$$= d_H\Big([t(\alpha+1)(x_l^\alpha(t))^2 + \int_0^t (t-s)x_l^\alpha(t),$$
$$t(3-\alpha)(x_r^\alpha(t))^2 + \int_0^t (t-s)x_r^\alpha(t)],$$
$$[t(\alpha+1)(y_l^\alpha(t))^2 + \int_0^t (t-s)y_l^\alpha(t),$$
$$t(3-\alpha)(y_r^\alpha(t))^2 + \int_0^t (t-s)y_r^\alpha(t)]\Big)$$
$$= t\max\{(\alpha+1)|(x_l^\alpha(t))^2 - (y_l^\alpha(t))^2|,$$
$$(3-\alpha)|(x_r^\alpha(t))^2 - (y_r^\alpha(t))^2|\}$$
$$+ \int_0^t (t-s)d_H([x_l^\alpha(s), x_r^\alpha(s)], [y_l^\alpha(s), y_r^\alpha(s)])$$
$$\leq 3T|x_r^\alpha(t) + y_r^\alpha(t)|$$
$$\times \max\{|x_l^\alpha(t) - y_l^\alpha(t)|, |x_r^\alpha(t) - y_r^\alpha(t)|\}$$
$$+ \frac{T^2}{2}\max\{|x_l^\alpha(t) - y_l^\alpha(t)|, |x_r^\alpha(t) - y_r^\alpha(t)|\}$$
$$= k_1 d_H([x(t)]^\alpha, [y(t)]^\alpha) + k_2 d_H([x(t)]^\alpha, [y(t)]^\alpha),$$

where k_1 and k_2 are satisfies the inequality in hypotheses (H1)-(H2), and also we have

$$d_H([g(x)]^\alpha, [g(y)]^\alpha)$$
$$= d_H(\sum_{k=1}^p C_k[x(t_k)]^\alpha, \sum_{k=1}^p C_k[y(t_k)]^\alpha)$$
$$\leq |\sum_{k=1}^p C_k| \max_k d_H([x(t_k)]^\alpha, [y(t_k)]^\alpha)$$
$$= L\, d_H([x(t_k)]^\alpha, [y(t_k)]^\alpha),$$

where L satisfies the inequality in hypothesis (H3).

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