비모의성 비용구조와 생산납기구간 환경에서의 동적롯사이징 모델
Dynamic Lot-Sizing Model with Production Time Windows under Nonspeculative Cost Structure

Hark-Chin Hwang
Department of Industrial Engineering, Chosun University,
375 Seosuk-Dong, Dong-Gu, Gwangju 501-759, South Korea
hchwang@chosun.ac.kr

Abstract

We consider dynamic lot-sizing model with production time windows where each of *n* demands has earliest due date and latest due date and it must be satisfied during the given time window. For the case of nonspeculative cost structure, an $O(n\log n)$ time procedure is developed and it is shown to run in $O(n)$ when demands come in the order of latest due dates.

1. Introduction

As the relationship between customers and suppliers is getting closer, it is not unusual that the supply contract is established based on interval of periods during which total agreed quantity should be delivered. To deal with this situation, lot-sizing models with *time windows* have been studied. In these models, we need to schedule in T planning horizon the demands i, $i = 1, 2, \ldots, n$, each having time window specified by earliest due date (EDD) and latest due date (LDD) between which it be satisfied. The application of the model with time windows can also be found in third party logistics and vendor managed inventory practices (Lee et al., 2001; Jaruphongsa et al. 2004s). There are two kinds of time windows: In the *production time window* case, each demand must be produced during its time window. In the other case of *delivery time window*, the constraint is relaxed that it is allowable to have a demand fulfilled out of its time window but with penalty of inventory holding or backlogging costs.

In the past production systems, being relied highly on forecasting, demands were aggregated by periods. That is, each demand's due date was given by single period. For this situation, Wagner and Whitin (1958) introduced the *classical* dynamic lot-sizing model to generate production schedules. Since then, there has been enormous study on this subject with various cost structures: in particular, nonspeculative costs, fixed plus linear costs, and concave costs (Aggarwal and Park, 1993; Wolsey, 1995; Brahimi et al., 2006). We note that nonspeculative cost is a special case of fixed plus linear cost where the unit production cost plus unit inventory cost of the current period is no less than that of the previous period. The consideration of (delivery) time windows in dynamic lot-sizing was first started by Lee et al. (2001) and then consideration of production time window has been followed by Dauzère-Pérès and Brahimi (2005). The original dynamic lot-sizing model with production time windows allows inventory: if we have production for a demand during its time window, the amount replenished is carried over to dispatch to customers in its last due date. However, as shown in Wolsey (2005), this model is equivalent to the revised model unallowing inventory. In the revised one, it is assumed that once a production occurs in a period, the produced amount is delivered just in the same period. Hence, this model can be thought of as a special case of the model with delivery time windows where inventory and backlogging penalty costs are set to infinite. In this paper, we will focus on the revised model by Wolsey (2005) and develop polynomial time optimal algorithms for the case.

The classical dynamic lot-sizing model with fixed plus linear cost structure can be solved in $O(T^2)$ using ordinary dynamic programming. To speed up such type of dynamic programming problems, three independent works of Aggarwal and Park (1993), Federgruen and Tzur (1991), Wagelmans et al. (1992) came out: it has been shown that the classical dynamic lot-sizing models with nonspeculative costs, fixed plus linear costs can be solved in time $O(T)$ and $O(T\log T)$, respectively. Furthermore, in Van Hoesel et al. (1994), they generalized the approaches in Federgruen and Tzur (1991), Wagelmans et al. (1992) using geometric techniques. In the case of delivery time windows with nonspeculative costs, Lee et al. (2001) provided $O(T^2)$ and $O(T^3)$ procedures for the case of backlogging unallowed and the other case of backlogging allowed, respectively. Recently, for the same model with backlogging, Hwang (2005) proposed a more efficient procedure with $O(\max\{T^2, nT\})$. To deal with a more general case of the fixed plus linear cost structure, Hwang and Jaruphongsa (2006) developed an $O(nT^3)$ algorithm.

For the model with production time windows, Dauzère-Pérès and Brahimi (2005) presented an $O(T^2)$ algorithm for nonspeculative costs and pseudopolynomial algorithm for the fixed plus linear cost structure. For this model with fixed plus linear cost structure, we can employ the algorithm of Hwang and Jaruphongsa (2006). For the concave cost structure, Veinott (1963) and Zangwill (1996) provided optimal algorithms in the classical dynamic lot-sizing model. However, until now on, no optimal algorithm is developed for the most general concave cost structure. For a special case that each time window does not overlay other windows, Hwang and Jaruphongsa (2004) provided an optimal algorithm.

In this paper, we consider dynamic lot-sizing model with production time windows. For the case when each period exhibits nonspeculative cost, an algorithm with complexity $O(n\log n)$ is devised. As shown later, when demands come in sorted order of LDD, it is proven to operate in $O(n)$.

In the next section, our model will be formally defined and known optimality properties be reviewed. The $O(n\log n)$ optimal procedure for nonspeculative costs is presented in Section 3. In the final section, we conclude this paper.

2. The Model and Basic Optimality Properties

We first introduce basic notations:

Parameters

- d_i : the required quantity for demand i for $i = 1, \ldots, n$.
- $[E_i, L_i]$: the time window of demand i for $i = 1, \ldots, n$ where demand i's EDD and LDD are denoted by E_i and L_i, respectively.
- p_t : the production function in period t. When the function is of fixed and linear cost structure, it is written as $K_t + p_t{\cdot}x$ where K_t is the setup cost and p_t is the unit production cost in period t. Under the nonspeculative costs, it holds that $p_t \geq p_{t+1}$.

Decision variables

- y_{it} : the amount dispatched in period t for demand i for $i = 1, \ldots, n$ and $t = 1, \ldots, T$.
- x_t : the amount replenished in period t for $t = 1, \ldots, T$.

The mathematical formulation of the problem is given by:

$$\text{Minimize} \sum_{t=1}^{T} p_t(x_t)$$

Subject to

$$\begin{aligned}
x_t - \textstyle\sum_{i=1}^{n} y_{it} = 0 & \qquad t = 1, \ldots, T \\
\sum_{t\in[E_i, L_i]} y_{it} = d_i & \qquad i = 1, \ldots, n \\
y_{it} \geq 0 & \qquad i = 1, \ldots, n \quad t \in [E_i, L_i] \\
y_{it} = 0 & \qquad i = 1, \ldots, n \quad t \notin [E_i, L_i] \\
x_t \geq 0 & \qquad t = 1, \ldots, T
\end{aligned}$$

Let's consider one of the most basic properties of the model with time windows. Though the following property on nonsplitting principle is proved for the delivery time window model with nonspeculative cost structure (2001), it is still applicable to our model.

Property 1: (*Nonsplitting Principle*) There is an optimal solution such that $y_{it} = d_i$ for some $t \in [E_i, L_i]$ for $i = 1, \ldots, n$.

This property implies that each demand can be satisfied by single dispatch. The dispatch period of demand i is denoted by $u(i) \in [E_i, L_i]$. Then, the following property (Wagner and Whitin, 1958; Zangwill, 1966) ensures that we can have an optimal solution with each demand being satisfied by a single replenishment.

Property 2: (*Planning Horizon Theorem*) There is an optimal solution such that between any two consecutive production periods τ_1 and τ_2, the replenishment in period τ_1 is used to cover the requirements $r(\tau_1), \ldots, r(\tau_2-1)$, where $r(t) = \sum_{1\leq i \leq n:u(i)=t} d_i$ for $t = 1, \ldots, T$.

Note that for each demand i, it has the same dispatch and replenishment period $u(i)$.

3. An Optimal Procedure with $O(n\log n)$

In nonspeculative cost structure, the production function $p_t(x)$ can be represented in the form of $K_t + p_t{\cdot}x$ where $p_{t-1} \geq p_t$ for all $t = 2, 3, \ldots, T$. By the constraint of production time window that each demand must be replenished within its time window, we know that any demand with EDD > t is satisfied later than t. Furthermore, any demand with LDD < t is replenished before t. Then the final question is about the demands *crossing* the period t? For these demands, we have the following property derived by Lee et al. (2001)

Property 3: Suppose that $p_{t-1} \geq p_t$ for all $t = 2, 3, \ldots, T$. Then, in an optimal solution, if we have a production in period t, then all the demands with their LDD $\geq t$ are replenished at or after the period t.

Before describing the optimal solution procedure, we first present necessary notations, which make it possible to arrange and group demands by EDD and LDD.

- $\alpha(i)$: the sorted list of demands in nondecreasing order of EDD so that $E_{\alpha(i)} \le E_{\alpha(i+1)}$ for $i = 1, 2, \ldots, n-1$.
- $\bar{\alpha}(t)$: the largest index i in the list α such that its corresponding demand $\alpha(i)$ has EDD equal to t, i.e., $E_{\alpha(i)} = t$ for $t = 1, 2, \ldots, T$. If no such $\bar{\alpha}(t)$ exists, we let $\bar{\alpha}(t) = \bar{\alpha}(t-1)$, where $\bar{\alpha}(0) = 0$. Then all the demands with EDD of t are the ones $\alpha(i)$ for $\bar{\alpha}(t-1) < i \le \bar{\alpha}(t)$.
- $\beta(i)$: the sorted list of demands in nondecreasing order of LDD so that $L_{\beta(i)} \le L_{\beta(i+1)}$ for $i = 1, 2, \ldots, n-1$.
- $\bar{\beta}(t)$: the largest index i in the list β such that its corresponding demand $\beta(i)$ has LDD equal to t, i.e., $L_{\beta(i)} = t$ for $t = 1, 2, \ldots, T$. If no such $\bar{\beta}(t)$ exists, we let $\bar{\beta}(t) = \bar{\beta}(t-1)$, where $\bar{\beta}(0) = 0$. Then all the demands with LDD of t are the ones $\beta(i)$, for $\bar{\beta}(t-1) < i \le \bar{\beta}(t)$.

Note that both the lists $\alpha(i)$ and $\beta(i)$ can be computed in $O(n\log n)$. Also, we can have $\bar{\alpha}(t)$ and $\bar{\beta}(t)$ for $t = 1, 2, \ldots, T$ in $O(n+T)$ using the lists $\alpha(i)$ and $\beta(i)$, respectively.

We need to define some further notations for intervals $[\lambda, \gamma]$, $1 \le \lambda < \gamma \le T$.

- $D(\lambda, \gamma)$: the set of demands j whose LDD is between λ and γ, i.e., $\lambda \le L_j < \gamma$. Then, we can denote by $D(1, \gamma)$ all the demands whose LDD is strictly less than γ.
- $C(\lambda, \gamma)$: the set of demands j crossing λ, whose LDD is between λ and γ, and EDD is no later than λ, i.e., $E_j \le \lambda \le L_j < \gamma$.

We let $a(\gamma)$ be the demand with the largest EDD among the demands in $D(1, \gamma)$. Ties are broken by choosing the demand with the smallest LDD among such demands. Also let $b(\gamma)$ be the demand with the largest LDD among the demands in $D(1, \gamma)$. Then, it is easy to see that the last production before γ, if it exists, occurs during the periods $E_{a(\gamma)}$, $E_{a(\gamma)}+1$, …, $L_{b(\gamma)}$. Then, we need to know how to obtain $a(\gamma)$ and $b(\gamma)$. Let $a(0) = b(0) = 0$ and suppose we are given $a(\gamma-1)$ and $b(\gamma-1)$. Then the demands $a(\gamma)$ and $b(\gamma)$ can be found as follows: If no demand exists with LDD equal to $\gamma-1$, then by definition we have that $a(\gamma) = a(\gamma-1)$ and $b(\gamma) = b(\gamma-1)$. Next consider the other case that demand exists with LDD equal to $\gamma-1$ and let demand i be of the smallest EDD among them. The demand i can be found by screening those demands $\beta(j)$, for $\bar{\beta}(\gamma-2) < j \le \bar{\beta}(\gamma-1)$. We note that the demand i is just $b(\gamma)$. Next, for $a(\gamma)$, if $E_i \le E_{a(\gamma)}$, then we have $a(\gamma) = a(\gamma-1)$. Otherwise, we have $a(\gamma) = i$. Repeating in this manner, we can find all the $a(1)$, $a(2)$, …, $a(T+1)$ and $b(1)$, $b(2)$, …, $b(T+1)$ just in time $O(n)$.

We let $d_{\lambda,\gamma}$ be the total sum of demands in the set of $D(\lambda, \gamma)$. Then, given $d_{1,\gamma}$, the cumulative sum of $d_{1,\gamma+1}$ can be found by the following recursion:

$$d_{1,\gamma+1} = d_{1,\gamma} + \sum_{\bar{\beta}(\gamma-1) < i \le \bar{\beta}(\gamma)} d_{\beta(i)}$$

Hence, we can see that all the $d_{1,\gamma}$ for $1 < \gamma \le T+1$ can be computed in time $O(n)$ by sweeping once the list β. Moreover, $d_{\lambda,\gamma}$ for the set $D(\lambda, \gamma)$ is immediately calculated using the following simple formula:

$$d_{\lambda,\gamma} = d_{1,\gamma} - d_{1,\lambda}.$$

By this time, we have developed necessary notations and now we would like to provide an optimal procedure. Let $f(\gamma)$ be the optimum cost for producing demands whose LDDs are strictly less than γ. When the last production occurs in period λ during the periods 1, 2, …, $\gamma-1$, the demands in $C(\lambda, \gamma)$ are produced at or after the period λ by Property 3. Furthermore, taking into account the fact that no production is scheduled during $\lambda+1$, $\lambda+2$, …, γ, the demands in $C(\lambda, \gamma)$ must be produced at the period λ. Since each demand must be satisfied within its time window, we know that there is no demand between the two periods λ and γ. That is, $C(\lambda, \gamma) = D(\lambda, \gamma)$. In addition, it must be the case that $E_{a(\gamma)} \le \lambda \le L_{b(\gamma)}$. Since the total amount replenished in period λ is the sum of demands in $C(\lambda, \gamma)$, i.e., $d_{\lambda,\gamma}$, the value of $f(\gamma)$ is given by $f(\gamma) = f(\lambda-1) + K_\lambda + p_\lambda d_{\lambda,\gamma}$. In general, $f(\gamma)$ is computed by the following recursion:

$$\begin{aligned} &f(0) = 0, \\ &f(\gamma) = 0, \text{ if } a(\gamma) = b(\gamma) = 0, \ 1 < \gamma \le T+1, \\ &f(\gamma) = \min_{E_{a(\gamma)} \le \lambda \le L_{b(\gamma)}} \{f(\lambda-1) + K_\lambda + p_\lambda d_{\lambda,\gamma}\}, \ 1 < \gamma \le T+1. \end{aligned} \tag{1}$$

Here, the optimal cost is $f(T+1)$, and it takes $O(T^2)$ to compute $f(T+1)$. However, it can be improved using known results for this type of dynamic programming: it can be solved in $O(T\log T)$ in general, and when the cost structure is nonspeculative it can be solved in a short time of $O(T)$ (Aggarwal and Park, 1993; Federgruen and Tzur, 1991; Wagelmans et al., 1992; Van Hoesel et al., 1994). In this paper, we apply the method in Van Hoesel et al. (1994) which utilizes geometric techniques. They provided a procedure operating in time $O(T\log T)$ for the type of the following dynamic programming:

$$\begin{aligned} &G(0) = 0, \\ &G(i) = A_i + \min_{1 \le j \le i}\{G(j-1) + B_j + C_j \cdot D_i\}, \ 1 \le i \le T. \end{aligned}$$

They also showed that if C_j and D_i are monotone, the dynamic programming can be solved in $O(T)$. Rewriting the main formula for $f(\gamma)$ in (1), we have, for $1 < \gamma \le T+1$,

$$f(\gamma) = \min_{E_{a(\gamma)} \le \lambda \le L_{b(\gamma)}} \{ f(\lambda - 1) + (K_\lambda - p_\lambda d_{1,\lambda}) + p_\lambda d_{1,\gamma} \}.$$

Here, γ has the role of i in the function G and λ corresponds to j. And, A_i is set to zero, and the terms $(K_\lambda - p_\lambda d_{1,\lambda})$, p_λ and $d_{1,\gamma}$ play the roles of B_j, C_j, and D_i, respectively. Moreover, p_λ and $d_{1,\gamma}$ are monotone because of the nonspeculative cost structure of p_λ and the fact that $d_{1,\gamma} \le d_{1,\gamma+1}$. Thus, our procedure in (1) runs in $O(T)$. Even though the running time is $O(T)$, the overall complexity for finding the optimal solution $f(T+1)$ is $O(n\log n)$, since we need to preprocess the values of $a(\gamma)$, $b(\gamma)$ and $d_{\lambda,\gamma}$, which requires the sorted list of β. We finally notice that if demands are given as a sorted list by LDD, we can find an optimal solution in $O(n)$.

4. Concluding Remarks

In this paper, we dealt with the dynamic lot-sizing model with production time windows during which demands must be satisfied. We showed that there exists an $O(n\log n)$ optimal algorithm for nonspeculative cost structure. Given sorted list of demands in latest due dates, it is also proven the algorithm runs in short time of $O(n)$.

References

Aggarwal, J. K. Park, Improved algorithms for economic lot-size problems, *Operations Research* 41 (1993), 549–571.

N. Brahimi, S. Dauzère-Pérès, N.M. Najid and A. Nordli. Single item lot sizing problems, *European Journal of Operational Research* 168 (2006), 1–16.

S. Dauzère-Pérès, N. Brahimi, N.M. Najid, and A. Nordli, Uncapacitated lot-sizing problems with time windows, Technical report, Ecole des Mines de Saint-Etienne (2005).

Federgruen, A., M. Tzur, A simple forward algorithm to solve general dynamic lot-sizing models with n periods in $O(n\log n)$ or $O(n)$ time, *Management Science* 37 (1991), 909–925.

H.-C. Hwang, W. Jaruphongsa, Dynamic Lot-Sizing Model for Major and Minor Demands, Working paper, Department of Industrial Engineering, Chosun University, South Korea (2004).

H.-C. Hwang, An Efficient Procedure for Dynamic Lot-Sizing Model Demand Time Windows, under revision in *Journal of Global Optimization* (2005).

H.-C. Hwang, W. Jaruphongsa, Dynamic lot-sizing model with demand time windows and speculative cost structure, *Operations Research Letters* 34 (2006) 251–256

W. Jaruphongsa, S. Çetinkaya, C.-Y. Lee, A Two-Echelon Inventory Optimization Model with Demand Time Window Considerations, *Journal of Global Optimization* 30 (2004), 347–366.

W. Jaruphongsa, S. Çetinkaya, C.-Y. Lee, Warehouse Space Capacity and Delivery Time Window Considerations in Dynamic Lot-Sizing for A Simple Supply Chain, International Journal of Production Economics 92 (2004), 169-180.

C.-Y. Lee, S. Çetinkaya, A. P. M. Wagelmans, Dynamic Lot-Sizing Model with Demand Time Windows, *Management Science* 47 (2001), 1384–1395.

Van Hoesel, A.P.M. Wagelmans and B. Moerman, Using geometric techniques to improve dynamic programming algorithms for the economic lot-sizing problem and extensions, European Journal of Operational Research 75 (1994), 312–331.

A.F. Veinott, Jr., Unpublished Class Notes, Program in Operations Research, Stanford University, Stanford, Calif (1963).

A.P.M. Wagelmans, S. Van Hoesel, A. Kolen, Economic lot-sizing: An $O(n\log n)$-algorithm that runs in linear time in the Wagner-Whitin case, *Operations Research* 40 (1992), S145–S156.

H.M. Wagner, T. M. Whitin, Dynamic version of the economic lot-size model, Management Science 5 (1958), 89–96.

LA.Wolsey, Progress with single-item lot-sizing, *Journal of Operational Research* 86 (1995), 395–401.

LA. Wolsey, Lot-sizing with production and delivery time windows, CORE DISCUSSION PAPER, Université catholique de Louvain (2005), available in http://www.core.ucl.ac.be/services/psfiles/dp05/dp2005_43.pdf

W.I. Zangwill, A deterministic multi-period production scheduling model with backlogging, *Management Science* 13 (1966), 105–119.