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# 퍼지수치 퍼지수 상의 쇼케이 거리측도에 관한 성질

# A note on the Choquet distance measures for fuzzy number-valued fuzzy numbers

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**요약**

구간치 퍼지집합은 Gorzalczang(1983)과 Turken(1986)에 의해 처음 제의되었다. 이를 토대로 Wang과 Li는 구간치 퍼지수에 관한 연산으로 일반화 하여 연구하였다. 최근에 홍(2002)는 왕과 리의 이론을 리만적분에 의해 구간치 퍼지집합상의 거리측도에 관한 연구를 하였다. 우리는 일반측도와 관련된 리만적분 대신에 퍼지측도와 관련된 쇼케이적분을 이용한 구간치 퍼지수 상의 쇼케이 거리측도를 연구하였다(2005). 본 논문에서는 퍼지수에서 퍼지수로의 쇼케이 거리측도를 정의하고 이와 관련된 성질들을 조사하였다.

**Abstract**

Interval-valued fuzzy sets were suggested for the first time by Gorzalczang(1983) and Turken(1986). Based on this, Wang and Li extended their operations on interval-valued fuzzy numbers. Recently, Hong(2002) generalized results of Wang and Li and extended to interval-valued fuzzy sets with Riemann integral. Using interval-valued Choquet integrals with respect to a fuzzy measure instead of Riemann integrals with respect to a classical measure, we studied some characterizations of interval-valued Choquet distance(2005). In this paper, we define Choquet distance measure for fuzzy number-valued fuzzy numbers and investigate some algebraic properties of them.

**Key words** : Fuzzy number-valued fuzzy number, Distance measure, Choquet integral.

## 1. Introduction

Interval-valued fuzzy sets were suggested for the first time by Gorzalczang(1983) and Turken(1986). Based on this, Wang and Li extended their operations on interval-valued fuzzy numbers. Recently, Hong(2002) generalized results of Wang and Li and extended to interval-valued fuzzy sets with Riemann integral. Using interval-valued Choquet integrals with respect to a fuzzy measure instead of Riemann integrals with respect to a classical measure, we studied some characterizations of interval-valued Choquet distance(2005). In this paper, we define Choquet distance measure for fuzzy number-valued fuzzy numbers and investigate some algebraic properties of them.

In section 2, we give preliminary definitions which are required in the following discussion. In section 3, we deal some properties of fuzzy number-valued fuzzy numbers. In section 4, using the Choquet integral with respect to fuzzy measure, we define a Choquet distance measures for fuzzy number-valued fuzzy numbers and investigate some properties of them.

## 2. Definitions and Preliminaries

At first, we introduce interval numbers and their basic operations (see

[4,5,6,7,10,13]). Throughout this paper, $R^+$ will denote the interval $[0, \infty)$,

$$I(R^+) = \{[a, b] \mid a, b \in R^+ \text{ and } a \leq b\}.$$

Then an element in $I(R^+)$ is called an interval number.

**Definition 2.1.** Let $[a_1, b_1], [a_2, b_2] \in I(R^+)$ and $k \in R^+$. We define

$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$
$[a_1, b_1] \cdot [a_2, b_2] = [a_1 \cdot a_2, b_1 \cdot b_2]$
$k[a_1, b_1] = [ka_1, kb_1]$,
$[a_1, b_1] \leq [a_2, b_2]$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$,
$[a_1, b_1] < [a_2, b_2]$ if and only if $[a_1, b_1] \leq [a_2, b_2]$ but $[a_1, b_1] \neq [a_2, b_2]$.

Then $(I(R^+), d_H)$ is metric space, where $d_H$ is the Hausdorff metric defined by
$d_H(A, B) = max\{sup_{x \in A} inf_{y \in B} \mid x - y \mid, sup_{y \in B} inf_{x \in A} \mid x - y \mid \}$.
for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, it is easily to show that for each pair $[a_1, b_1], [a_2, b_2] \in I(R^+)$,
$d_H([a, b], [c, d]) = max\{ \mid a - c \mid, \mid b - d \mid \}$

We note that $\leq$ is called an order of interval numbers and that $[a, b] \subset [c, d]$ means $[a, b]$ is subset of $[c, d]$.

**Definition 2.2.** A fuzzy number is a fuzzy set $u$ on $R^+$, satisfying the following conditions:
(i) (normality) $u(x) = 0$ for some $x \in R^+$,
(ii) (convexity) for arbitrary $\lambda \in (0, 1]$, $[u]^\lambda = \{x \in R^+ \mid u(x) \geq \lambda\} \in I(R^+)$
and (iii)
$[u]^0 = \{x \in R^+ \mid u(x) > 0\} \in I(R^+)$.

Let $F(R^+)$ denote the set of all fuzzy numbers. we define for each pair $u, v \in F(R^+)$ and $k \in R^+$,

$[u + v]^\lambda = [u]^\lambda + [v]^\lambda$,
$[ku]^\lambda = k[u]^\lambda$,
$u \leq v$ if and only if $[u]^\lambda \leq [v]^\lambda$
for all $\lambda \in [0, 1]$.

**Definition 2.3.** Let $A \in F(R^+)$. Then $A$ is called an interval convex fuzzy set, if for any $x, y \in R$ and $\lambda \in [0, 1]$, we have

$$A(\lambda x + (1 - \lambda) y) \geq A(x) \wedge A(y).$$

**Definition 2.4.** Let $A, B \in F(R^+)$ and $\bullet \in \{+, -, \cdot, \div\}$. We define their extended operations to

$$(A \bullet B)(z) = \vee_{z = x \bullet y} (A(x) \wedge B(y)).$$

For each $[\lambda_1, \lambda_2] \in [I]^+$, we write
$A_{[\lambda_1, \lambda_2]} \bullet B_{[\lambda_1, \lambda_2]}$
$= \{x \bullet y : x \in A_{[\lambda_1, \lambda_2]}, y \in B_{[\lambda_1, \lambda_2]}\}$

**Definition 2.5.**([9,10,11,12]) (1) A fuzzy measure on a measurable space $(X, \Omega)$ is an extended real-valued function $\mu : \Omega \to [0, \infty]$ satisfying
(i) $\mu(\phi) = 0, \mu(X) = 1$
(ii) whenever $A, B \in \Omega, A \subset B$, then $\mu(A) \leq \mu(B)$.
(2) $\mu$ is said to be continuous from below it for every increasing sequence $\{A_n\} \subset \Omega$ of measurable sets, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

(3) $\mu$ is said to be continuous from above it for every decreasing sequence $\{A_n\} \subset \Omega$ of measurable sets, we have

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

(4) if $\mu$ is said to be continuous from above and continuous from below, it is said to be continuous.

Recall that a function $f : X \to [0, \infty]$ is said to be measurable if $\{x \mid f(x) > \alpha\} \in \Omega$ for all $\alpha \in (-\infty, \infty)$.

**Definition 2.6.**([9,10,11,12]) (1) The Choquet integral of a measure $\mu$ is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r)\, dr$$

where $\mu_f(r) = \mu(\{x \mid f(x) > r\})$ and the integral on the right-hand side is an ordinary one.
(2) A measurable function $f$ is called integrable if the choquet integral of $f$ can be defined and its value is finite.

**Definition 2.7.** ([9,10,11,12]) Let $f, g$ be measurable nonnegative functions. We say

that $f$ and $g$ are comonotonic, in symbol $f \sim g$ if and only if
$f(x) < f(x') \Rightarrow g(x) \leq g(x')$ for all $x, x' \in X$.

**Theorem 2.8.**([5,9,11,12]) Let $f, g, h$ be measurable functions. Then we have
(1) $f \sim f$,
(2) $f \sim g \;\Rightarrow\; g \sim f$,
(3) $f \sim a$ for all $a \in R^+$,
(4) $f \sim g$ and $f \sim h \Rightarrow f \sim (g+h)$.

**Theorem 2.9.**([5,9,11,12]) Let $f, g$ be nonnegative measurable functions.
(1) If $f \leq g$, then $(C)\int f d\mu \leq (C)\int g d\mu$.
(2) If $f \sim g$ and $a, b \in R^+$, then
$$(C)\int (af+bg) d\mu = a(C)\int f d\mu + b(C)\int g d\mu.$$
(3) If $(f \vee g)(x) = f(x) \vee g(x)$, then
$$(C)\int f \vee g \, d\mu \geq (C)\int f d\mu \vee (C)\int g d\mu,$$
and
If $(f \wedge g)(x) = f(x) \wedge g(x)$, then
$$(C)\int f \wedge g \, d\mu \leq (C)\int f d\mu \wedge (C)\int g d\mu$$
for all $x \in X$.

**Theorem 2.10.**([9,10]) (1) If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions, then we have
$$(C)\int \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} (C)\int f_n d\mu.$$
(2) If $\{f_n\}$ is a decreasing sequence of nonnegative measurable functions and $f_1$ is Choquet integrable, then we have
$$(C)\int \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} (C)\int f_n d\mu.$$

## 3. Fuzzy number-valued fuzzy numbers

In this section, we will define various concepts associated with fuzzy number-valued fuzzy number.

**Definition 3.1.** Let $A, B \in F F^*(R^+)$ and $\bullet \in \{+, -, \cdot, \div\}$. Then we have
$$(A \bullet B)(z) = [(A_- \bullet B_-)(z),\ (A^- \bullet B^-)(z)]$$

**Corollary 3.2.** [3,12] Let $A, B \in F F^*(R^+)$. Then $A+B,\ A-B,\ A \cdot B,\ A \div B \in FF^*(R)$.

The following to important results immediate as an application of Theorem 3.2 and commutatively and associativity of fuzzy numbers under $+$ and $\bullet$ (see[3, 12]).

**Theorem 3.3.** Let $A, B \in F F^*(R^+)$. Then $A \bullet B = B \bullet A$ where $\bullet \in \{+, \cdot\}$.

**Theorem 3.4.** Let $A,B,C \in F(R^+)F F^*(R^+)$. Then $(A \bullet B) \bullet C = A \bullet (B \bullet C)$ where $\bullet \in \{+, \cdot\}$.

## 4. Main Results

At first, we define Choquet distance measures between interval-valued fuzzy numbers.

**Definition 4.1.** For arbitrary interval-valued fuzzy numbers $A, B \in IF^*(R)$, the quantity
$$D_c(A, B) = (C)\int d_H(A(x), B(x)) d\mu(x)$$
$$= \int_0^{\infty} \mu\{x \mid d_H(A(x), B(x)) > \alpha\} d\alpha$$
is the Choquet distance measure between $A$ and $B$, where $d_H$ is the Hausdorff metric between $A(x)$ and $B(x)$ which is defined as
$$d_H(A(x), B(x))$$
$$= d_H(A_-(x), B_-(x)) \vee d_H(A^-(x), B^-(x))$$
since $A_-(x)$ and $A^-(x)$ the lower and the upper endpoint of $A(x)$,
$$A(x) = [A_-(x),\ A^-(x)].$$

Now we define Choquet distance measures between fuzzy number-valued fuzzy numbers.

**Definition 4.2.** For arbitrary fuzzy number-valued fuzzy numbers $A, B \in F(R^+)$, the quantity
$$\Delta_1(A, B) = \int_0^1 D_C(A_\lambda, B_\lambda) d\lambda$$
where $A_\lambda,\ B_\lambda \in IF^*(R^+)$ and $D_C(A_\lambda, B_\lambda)$ is distance of $A_\lambda$ and $B_\lambda$.

**Theorem 4.3.** Let $A, B \in F F^*(R^+)$. Then we have
$$\Delta_1 : FF^*(R^+) \times F F^*(R^+) \rightarrow [0, \infty]$$
is pseudo-metric.

**Theorem 4.4.** Let $A, B \in F(R^+)$ and $A, B$ are continuous. Then $\Delta_1(A, B) = 0$ if and only if $A = B$ $\mu - a.e.$

**Theorem 4.5.** Let $\{A_n\}$ is an increasing sequence of fuzzy number-valued fuzzy numbers in $FF^*(R^+)$ and for each $x \in R^+$, $d_H - \lim_{n \to \infty} A_n(x) = A(x)$. Then we have

$$\lim_{n \to \infty} \Delta_1(A_n, A) = 0.$$

**Theorem 4.6.** Let $A \in FF^*(R^+)$. If $B \in FF^*(R^+)$ and $A_n \in FF^*(R^+)$ for $n = 1, 2, 3, \cdots$.Then
$\Delta_1(\vee A_n, B) \geq \vee \Delta_1(A_n, B)$ and
$\Delta_1(\wedge A_n, B) \leq \wedge \Delta_1(A_n, B)$.

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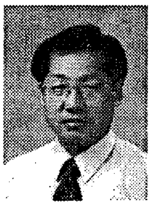

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