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Comparison of Parameter Estimation Methods in A Kappa Distribution*

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Abstract

This paper deals with the comparison of parameter estimation methods in a 3-parameter Kappa distribution which is sometimes used in flood frequency analysis. The method of moment estimation(MME), L-moment estimation(L-ME), and maximum likelihood estimation(MLE) are applied to estimate three parameters. The performance of these methods are compared by Monte-carlo simulations. Especially for computing MME and L-ME, three dimensional nonlinear equations are simplied to one dimensional equation which is calculated by the Newton-Raphson iteration under constraint. Based on the criterion of the mean squared error, the L-ME is recommended to use for small sample size ($n \leq 100$) while MLE is good for large sample size.

Keywords : MME, L-ME, MLE, equivariance, quantile estimation, Kappa distribution.

1. Introduction

A family of positively skewed distributions, called kappa distribution, has been introduced by Mielke[8], and Mielke and Johnson[7]. The kappa distribution has received limited attention from the hydrologic community for analyzing the precipitation data and the stream ow data.

In this paper, a three-parameter kappa distribution is considered. The method of moments, the maximum likelihood, and L-moments methods are presented for estimating its parameters and compared for estimating the upper quantiles. A comparison of its performance are made by Monte-Carlo simulation.

2. Kappa distribution and moments

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The cumulative distribution function(CDF) of the three-parameter kappa distribution(KD3) is

$$F(x)=\left(\frac{x-\mu}{\beta}\right)\left[\alpha+\left(\frac{x-\mu}{\beta}\right)^{\alpha}\right]^{-1/\alpha}, \tag{1}$$

with bounds of $\alpha>0$, $\beta>0$, $\mu\leq\min(x)$. Here μ is a location parameter, β is a scale parameter, and α is a shape parameter. The quantile function is

$$x(F)=\mu+\beta\left(\frac{\alpha F^{\alpha}}{1-F^{\alpha}}\right)^{1/\alpha}. \tag{2}$$

Compared to the gamma distribution and the lognormal distribution which has been used to describe the precipitation data, the kappa distribution has closed algebraic expressions for CDF and quantile functions. Thus it provides easier calculations.

The first moment of KD3 is obtained as follows:

$$\begin{aligned}E(x)=\int_0^1 x(F)dF &= \int_0^1\left[\mu+\left(\frac{\alpha F^{\alpha}}{1-F^{\alpha}}\right)^{1/\alpha}\right]dF\\ &=\mu+\int_0^1 \beta\alpha^{\frac{\alpha-1}{\alpha}}U^{\frac{2}{\alpha}-1}(1-U)^{\frac{\alpha-1}{\alpha}-1}dU\\ &\quad\text{(by letting } U=F^{\alpha})\\ &=\mu+\beta\alpha^{-\frac{\alpha-1}{\alpha}}B\left(\frac{2}{\alpha},\frac{\alpha-1}{\alpha}\right),\ \alpha>1,\end{aligned} \tag{3}$$

where $B(x,y)$ is the beta function. Similar calculations give the r-th moments in general;

$$\begin{aligned}\mu_r &= E(X^r)=\int_0^1[x(F)]^r dF\\ &=\sum_{i=0}^{r}\binom{r}{i}\beta^i\mu^{r-i}h_i,\ \alpha>r,\end{aligned} \tag{4}$$

where $h_i=h_i(\alpha)$

$$=\alpha^{-\frac{\alpha-1}{\alpha}}B\left(\frac{i+1}{\alpha},\frac{\alpha-i}{\alpha}\right),\ \alpha>i,\ h_0=h_0(\alpha)=1. \tag{5}$$

The L-moments has been introduced by Hosking[2], and used widely in hydrologic community([10][5][13][1][14], for example). Since it is now a standard method in analyzing skewed data (e.g. extreme rainfall data, flood frequency data, extreme wind speed data, etc.), the details are omitted here (see [2][10][15]). The first three L-moments (λ_r) of KD3 are obtained using the relations with the probability weighted moments [12];

$$\lambda_1 = \mu + \beta k_1, \tag{6}$$

$$\lambda_2 = \beta[2k_2 - k_1], \tag{7}$$

$$\lambda_3 = \beta[6k_3 - 6k_2 + k_1], \tag{8}$$

where $k_r = k_r(\alpha) = \alpha^{-\frac{\alpha-1}{\alpha}} B\left(\frac{r+1}{\alpha}, \frac{\alpha-1}{\alpha}\right), \quad \alpha > 1.$

3. Estimation Methods

3.1 The method of moments estimation

Let m_r denote the sample moments. Then the method of moments estimates (MME) are obtained by equating first three moments to the corresponding sample moments as follows;

$$\mu_1 = \mu + \beta h_1 = m_1 \tag{9}$$

$$\mu_2 = \mu^2 + 2\mu\beta h_1 + \beta^2 h_2 = m_2 \tag{10}$$

$$\mu_3 = \mu^3 + 3\mu^2\beta h_1 + 3\mu\beta^2 h_2 + \beta^3 h_3 = m_3, \quad \alpha > 3. \tag{11}$$

Now by observing that the following quantity

$$\frac{(\mu^2 - \mu_1^2)^{3/2}}{\mu_3 - \mu_1^3 - 3\mu_1(\mu_2 - \mu_1^2)} = \frac{(h^2 - h_1^2)^{3/2}}{3h_1^3 - 3h_1 h_2 + h_3} \tag{12}$$

is only the function of α, the MME of α is obtained by equating (12) to the corresponding function of sample moments. Then this equation is solved by one-dimensional Newton-Raphson method under the constraint $\alpha > 3$. And estimates of other parameters are obtained directly from (9) and (10);

$$\hat{\beta} = \frac{m_2 - m_1^2}{h_2(\hat{\alpha}) - h_1^2(\hat{\alpha})'} \tag{13}$$

$$\hat{\mu} = m_1 - \hat{\beta} h_1(\hat{\alpha}). \tag{14}$$

Now denote (12) be $g(\alpha)$, then the first derivative of $g(\alpha)$ with respect to α which is needed in Newton-Raphson routine is calculated as follows:

$$\frac{dg(\alpha)}{d\alpha} = \frac{3}{2} \cdot \frac{h_2' - 2h_1 h_1'}{h_2 - h_1^2} - \frac{6h_1^2 h_1' - 3h_2 h_1' - 3h_1 h_2' + h_3'}{2h_1^3 - 3h_1 h_2 + h_3} g(\alpha) \tag{15}$$

where $h_i' = \frac{dh_i(\alpha)}{d\alpha}$

$$= -\frac{1}{\alpha} + \frac{i(1-\ln\alpha)}{\alpha^2} + \Psi\left(\frac{i+1}{\alpha}\right) + \Psi\left(\frac{\alpha-i}{\alpha}\right) - \Psi\left(\frac{\alpha+1}{\alpha}\right)h_i, \tag{16}$$

and $\Psi(x) = \frac{d}{dx}\log\Gamma(x)$ is the digamma function.

3.2 The method of L-moments estimation

Let l_r denote the r-th sample L-moments. Then the method of L-moments estimates (L-ME) are obtained by equating first three L-moments to the corresponding sample L-moments. Now observing that the so-called L-skewness

$$\frac{\lambda_3}{\lambda_2} = \frac{6k_3(\alpha) - 6k_2(\alpha) + k_1(\alpha)}{2k_2(\alpha) - k_1(\alpha)} \tag{17}$$

is only the function of α , the L-ME of α is obtained by equating (17) to the corresponding function of sample moments (i.e. the sample L-skewness l_3/l_2). This equation is solved by one-dimensional Newton-Raphson method under the constraint $\alpha > 1$. Then estimates of other parameters are obtained directly from (6) and (7);

$$\hat{\beta} = \frac{l_2}{2k_2(\hat{\alpha}) - k_1^2(\hat{\alpha})'} \tag{18}$$

$$\hat{\mu} = l_1 - \hat{\beta}k_1(\hat{\alpha}). \tag{19}$$

Now denote (17) be $f(\alpha)$, then the first derivative of $f(\alpha)$ with respect to α which is needed in Newton-Raphson routine is calculated as follows:

$$\frac{df(\alpha)}{d\alpha} = \left(\frac{6k_3' - k_2' + k_1'}{6k_3 - k_2 + k_1'} - \frac{2k_2' - k_1'}{2k_2 - k_1}\right)f(\alpha), \tag{20}$$

where $k_i' = \frac{dk_i(\alpha)}{d\alpha}$

$$= -\frac{1}{\alpha} + \frac{1-\ln\alpha}{\alpha^2} + \Psi\left(\frac{i+1}{\alpha}\right) + \Psi\left(\frac{\alpha-1}{\alpha}\right) - \Psi\left(\frac{\alpha+i}{\alpha}\right)k_i. \tag{21}$$

Note that the 3-dimensional system of equations problems for MME and L-ME are reduced to one dimensional equation problems by simple calculations, which lead easier computation for the estimations.

3.3 The maximum likelihood estimation

For given observations $(x_1, x_2, \cdots, x_n)$, the negative log-likelihood function $(\lambda = -\ln L(\mu, \alpha, \beta))$ of (μ, α, β) is

$$\lambda = \frac{\alpha+1}{\alpha} \sum_{i=1}^{n} \ln\left[\alpha + \left(\frac{x_i - \mu}{\beta}\right)^{\alpha}\right] - n\ln\left(\frac{\alpha}{\beta}\right), \tag{22}$$

$$\alpha > 0, \quad \beta > 0, \quad \mu \le \min(x).$$

The maximum likelihood estimates(MLE) are obtained by minimizing (22) with respect to parameters. Note that the MLE of μ is $\min(x)$ because it minimizes (22) for any fixed values of α and β.

Thus the MLE of α and β and are obtained from the adjusted likelihood:

$$-\ln L(\alpha, \beta) = \frac{\alpha+1}{\alpha} \sum_{i=2}^{n} \ln\left[\alpha + \left(\frac{x_{(i)} - \hat{\mu}}{\beta}\right)^{\alpha}\right] - n\ln\left(\frac{\alpha}{\beta}\right), \tag{23}$$

where $x_{(i)}$ denote i-th observation ordered ascendingly, and $\hat{\mu} = \min(x) = x(1)$. Since no explicit minimizer is possible, this function is numerically minimized by an iterative algorithm (a quasi-Newton routine). The first derivatives of (23) with respect to α and β which are needed in a quasi-Newton routine are calculated as follows:

$$\frac{\partial}{\partial \alpha}(-\ln L) = -\frac{1}{\alpha^2} \sum_{i=2}^{n} \ln[\alpha + y_i^{\alpha}] - \frac{n}{\alpha} + \frac{\alpha+1}{\alpha} \sum_{i=2}^{n} \frac{1 + y_i^{\alpha} \ln y_i}{\alpha + y_i^{\alpha}}, \tag{24}$$

$$\frac{\partial}{\partial \beta}(-\ln L) = -\frac{\alpha+1}{\beta} \sum_{i=2}^{n} \frac{y_i^{\alpha}}{\alpha + y_i^{\alpha}} + \frac{n}{\beta}, \tag{25}$$

where $y_i = (x_{(i)} - \hat{\mu})/\beta$.

4. Performance Evaluation

In order to evaluate the performance of these estimators, several parent distributions (K3D itself plus the 4-parameter Kappa[7] and Wakeby distributions[4]) are used. Sample sizes are 15, 30, 50, 75, 100, 150, 200, 300. The number of Monte-Carlo replication is 1000. The parameters μ and β are set to 0 and 1 respectively in generating Monte-Carlo samples, because those are location and scale equivariant.

By noting that the upper quantile estimates are more useful than the parameter estimates in real hydrologic applications of K3D, the squared error $(x_q - \widehat{x_q})^2$ has been used as a criterion for the performance evaluation, where $q=0.95, 0.99$ are considered. Here the x_q is the true q-quantile computed from a given true parameters, and $\hat{x}_q$ is the estimated q-quantile computed from the estimated K3D.

Based on the criterion of the mean squared error, the L-ME (or MME) is recommended to use for small sample size ($n \leq 100$) while MLE is good for large sample size. The details of this simulation study is omitted here.

5. Summary and Discussion

A comparison of parameter estimation methods was done in a 3-parameter Kappa distribution which is sometimes used in flood frequency analysis. The method of moment estimation (MME), L-moment estimation (L-ME), and maximum likelihood estimation (MLE) are applied to estimate three parameters. The performance of these methods are compared by Monte-carlo simulations. Especially for computing MME and L-ME, three dimensional nonlinear equations are simplied to one dimensional equation which is calculated by the Newton-Raphson iteration under constraint.

A Monte-Carlo simulation shows that the LME is recommended to use for small sample size ($n \leq 100$) while MLE is good for large sample size, based on the criterion of the mean squared error.

We think the three-parameter kappa distribution can be used for analyzing data from wider research area such as the (extreme) precipitation data, the (extreme) wind speed data and the (extreme) stream ow data.

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