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Estimation for the Rayleigh Distribution Based on Multiply Type-II Censored Sample

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Abstract

In this paper, we derive several approximate maximum likelihood estimators of the scale and location parameters in the Rayleigh distribution based on multiply Type-II censored samples. We compare the proposed estimators in the sense of the mean squared error for various censored samples.

Keywords : Approximate maximum likelihood estimator, Multiply Type-II censored sample, Rayleigh distribution

1. Introduction

The random variable X has the Rayleigh distribution if it has a probability density function (pdf) of the form

$$f(x;\theta,\sigma)=\frac{x-\theta}{\sigma^2}\exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\},\quad x\geq\theta,\ \sigma>0 \tag{1.1}$$

and the cumulative distribution function (cdf) of the form

$$F(x;\theta,\sigma)=1-\exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\},\quad x\geq\theta,\ \sigma>0, \tag{1.2}$$

where θ and σ are the location and the scale parameters, respectively.

Balakrishnan (1989) proposed the approximate maximum likelihood estimator (AMLE) of the scale parameter in the Rayleigh distribution with censoring. Cohen (1991) obtained the maximum likelihood estimators (MLEs) of the scale parameter in singly right censored and truncated samples from the Rayleigh distribution.

Multiply Type-II censoring is a generalization of Type-II censoring. Compared to Type-II censoring, multiply Type-II censoring is more general and mathematically and numerically much more complicated censoring scheme. Kong and Fei (1996) discussed the limit theorems for the maximum likelihood estimate

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under general multiply Type-II censoring. Upadhyay et al. (1996) considered the estimation for the exponential distribution under multiply Type-II censoring. Kang (2005) derived the AMLEs of the scale parameter and location parameter in the extreme value distribution based on multiply Type-II censored samples. Kang and Lee (2005) derived the AMLEs of the scale and location parameters in the two-parameter exponential distribution based on multiply Type-II censored samples. They also obtained the moments of the proposed estimators. Kang and Park (2005) derived the AMLEs for the location and scale parameters that are explicit function of order statistics in the exponentiated exponential distribution. They also compared the proposed estimators in the sense of the mean squared error (MSE) for various censored samples.

Recently, Han and Kang (2006) derived some AMLEs of the scale parameter when the location parameter is known and also derived an AMLE of the location parameter when the scale parameter is known in the Rayleigh distribution under multiply Type-II censoring.

In this paper, we derive the AMLEs of the scale parameter σ and the location parameter θ in the two-parameter Rayleigh distribution with multiply Type-II censoring by the approximate maximum likelihood estimation method. We also compare the proposed estimators in the sense of the MSE for various censored samples.

2. Approximate Maximum Likelihood Estimators

Let

$$X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_s:n} \tag{2.1}$$

be the multiply Type-II censored sample, where $1 \leq a_1 < a_2 < \ldots < a_s \leq n$ and $X_{1:n}, \cdots, X_{n:n}$ are order statistics of $X_1, \cdots, X_n$.

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) is given by

$$\begin{aligned} L = \frac{1}{\sigma^s} \frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!} [F(Z_{a_1:n})]^{a_1 - 1} [1 - F(Z_{a_s:n})]^{n-a_s} \\ \times \prod_{j=1}^{s} f(Z_{a_j:n}) \left[\prod_{j=2}^{s} [F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1} \right], \end{aligned} \tag{2.2}$$

where $Z_{i:n} = (X_{i:n} - \theta)/\sigma$, $f(z) = ze^{-z^2/2}$ and $F(z) = 1 - e^{-z^2/2}$ are the pdf and the cdf of the standard Rayleigh distribution, respectively.

From the equation (2.2), we obtain the likelihood equations as follows;

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\left[2s+(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n}-(n-a_s)Z_{a_s:n}^2-\sum_{j=1}^{s}Z_{a_j:n}^2\right.$$
$$\left.+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\right] \tag{2.3}$$
$$=0$$

and

$$\frac{\partial \ln L}{\partial \theta} = -\frac{1}{\sigma}\left[(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}-(n-a_s)Z_{a_s:n}+\sum_{j=1}^{s}\frac{1}{Z_{a_j:n}}-\sum_{j=1}^{s}Z_{a_j:n}\right.$$
$$\left.+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})-f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\right] \tag{2.4}$$
$$=0.$$

Since these likelihood equations are very complicated, the equations (2.3) and (2.4) do not admit explicit solutions for σ and θ. So we need some approximate likelihood equations which give explicit solutions.

Han and Kang (2006) obtained some approximations by Taylor series expansion as follows;

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n} \simeq \alpha_1+\beta_1 Z_{a_1:n} \tag{2.5}$$

$$\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{1j}+\beta_{1j}Z_{a_j:n}+\gamma_{1j}Z_{a_{j-1}:n} \tag{2.6}$$

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} \simeq \alpha_2+\beta_2 Z_{a_1:n} \tag{2.7}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n} \tag{2.8}$$

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n} \tag{2.9}$$

$$\frac{1}{Z_{a_j:n}} \simeq \kappa_j+\delta_j Z_{a_j:n} \tag{2.10}$$

$$\frac{f(Z_{a_j:n})-f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{4j}+\beta_{4j}Z_{a_j:n}+\gamma_{4j}Z_{a_j:n}, \tag{2.11}$$

where

$$\xi_i = F^{-1}(p_i) = \left[-2\ln(1-p_i)\right]^{1/2}$$
$$p_i = \frac{i}{n+1},\ q_i = 1-p_i$$
$$\alpha_1 = \frac{-\xi_{a_1}^2}{p_{a_1}}\left[f'(\xi_{a_1})-\frac{f^2(\xi_{a_1})}{p_{a_1}}\right]$$

$$\beta_1 = \frac{1}{p_{a_1}}\left[f(\xi_{a_1}) + \xi_{a_1} f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\alpha_{1j} = K^2 - \frac{\xi^2_{a_j} f'(\xi_{a_j}) - \xi^2_{a_{j-1}} f'(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}$$

$$\beta_{1j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1-K) f(\xi_{a_j}) + \xi_{a_j} f'(\xi_{a_j})\right]$$

$$\gamma_{1j} = -\frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1-K) f(\xi_{a_{j-1}}) + \xi_{a_{j-1}} f'(\xi_{a_{j-1}})\right]$$

$$K = \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}$$

$$\alpha_2 = \frac{1}{p_{a_1}}\left[f(\xi_{a_1}) - \xi_{a_1} f'(\xi_{a_1}) + \frac{f^2(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\beta_2 = \frac{1}{p_{a_1}}\left[f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}}\right]$$

$$\alpha_{2j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1+K) f(\xi_{a_j}) - \xi_{a_j} f'(\xi_{a_j})\right]$$

$$\beta_{2j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[f'(\xi_{a_j}) - \frac{f^2(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}}\right]$$

$$\gamma_{2j} = \frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{(p_{a_j} - p_{a_{j-1}})^2}$$

$$\alpha_{3j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1+K) f(\xi_{a_{j-1}}) - \xi_{a_{j-1}} f'(\xi_{a_{j-1}})\right]$$

$$\beta_{3j} = -\frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{(p_{a_j} - p_{a_{j-1}})^2} = -\gamma_{2j}$$

$$\gamma_{3j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[f'(\xi_{a_{j-1}}) + \frac{f^2(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}\right]$$

$$\kappa_j = 2/\xi_{a_j} \quad \delta_j = -1/\xi^2_{a_j} \quad \alpha_{4j} = \alpha_{2j} - \alpha_{3j}$$

$$\beta_{4j} = \beta_{2j} - \beta_{3j}, \quad \text{and} \quad \gamma_{4j} = \gamma_{2j} - \gamma_{3j}.$$

Now making use of the approximate expressions in (2.5), (2.6), (2.7), (2.10), and (2.11), we may approximate the likelihood equations of (2.3) and (2.4) as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq -\frac{1}{\sigma}\Bigg[2s + (a_1 - 1)(\alpha_1 + \beta_1 Z_{a_1:n}) - (n - a_s) Z^2_{a_s:n} - \sum_{j=1}^{s} Z^2_{a_j:n} \\ &\quad + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n})\Bigg] \\ &= 0\end{aligned} \tag{2.12}$$

and

$$\frac{\partial \ln L}{\partial \theta} \simeq -\frac{1}{\sigma}\Bigg[(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})-(n-a_s)Z_{a_s:n}+\sum_{j=1}^{s}(\kappa_j+\delta_j Z_{a_j:n}) \\ -\sum_{j=1}^{s} Z_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{4j}+\beta_{4j} Z_{a_j:n}+\gamma_{4j} Z_{a_{j-1}:n})\Bigg] \\ =0. \tag{2.13}$$

Upon solving the equations (2.12) and (2.13) for σ and θ, we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_1=\frac{-B_1+\sqrt{B_1^2-4A_1C_1}}{2A_1} \tag{2.14}$$

and

$$\hat{\theta}_1 = M_1\hat{\sigma}_1+M_2, \tag{2.15}$$

where

$$A_1= 2s+(a_1-1)(\alpha_1-\beta_1 M_1)-(n-a_s+s)M_1{}^2 \\ +\sum_{j=2}^{s}(a_j-a_{j-1}-1)\big(\alpha_{1j}-\beta_{1j}M_1-\gamma_{1j}M_1\big)$$

$$B_1= (a_1-1)\beta_1(X_{a_1:n}-M_2)+2(n-a_s)M_1(X_{a_s:n}-M_2)+2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2) \\ +\sum_{j=2}^{s}(a_j-a_{j-1}-1)\big[\beta_{1j}(X_{a_j:n}-M_2)+\gamma_{1j}(X_{a_{j-1}:n}-M_2)\big]$$

$$C_1=-(n-a_s)(X_{a_s:n}-M_2)^2-\sum_{j=1}^{s}(X_{a_j:n}-M_2)^2$$

$$A= (a_1-1)\alpha_2+\sum_{j=1}^{s}\kappa_j+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_{4j}$$

$$B= (a_1-1)\beta_2 X_{a_1:n}-(n-a_s)X_{a_s:n}+\sum_{j=1}^{s}\delta_j X_{a_j:n}-\sum_{j=1}^{s}X_{a_j:n} \\ +\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}X_{a_j:n}+\gamma_{4j}X_{a_{j-1}:n})$$

$$C= (a_1-1)\beta_2-(n-a_s)+\sum_{j=1}^{s}\delta_j-s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}+\gamma_{4j})$$

$$M_1=\frac{A}{C},\ M_2=\frac{B}{C}.$$

Second, making use of the approximate expressions in (2.6), (2.7), (2.10), and (2.11), we may approximate the likelihood equation of (2.3) as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq -\frac{1}{\sigma}\Big[2s+(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})Z_{a_1:n}-(n-a_s)Z_{a_s:n}^2-\sum_{j=1}^{s} Z_{a_j:n}^2 \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}+\beta_{1j}Z_{a_j:n}+\gamma_{1j}Z_{a_{j-1}:n})\Big] \\ &=0.\end{aligned} \tag{2.16}$$

Upon solving the equations (2.16) and (2.13) for σ and θ, we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_2=\frac{-B_2+\sqrt{B_2^2-4A_2C_2}}{2A_2} \tag{2.17}$$

and

$$\hat{\theta}_2 = M_1\hat{\sigma}_2+M_2, \tag{2.18}$$

where

$$\begin{aligned}A_2 &= 2s-(a_1-1)M_1(\alpha_2-\beta_2M_1)-(n-a_s+s)M_1^2 \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\alpha_{1j}-\beta_{1j}M_1-\gamma_{1j}M_1\right]\end{aligned}$$

$$\begin{aligned}B_2 &= (a_1-1)(\alpha_2-2\beta_2M_1)(X_{a_1:n}-M_2)+2(n-a_s)M_1(X_{a_s:n}-M_2) \\ &\quad +2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2)+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\beta_{1j}(X_{a_j:n}-M_2)+\gamma_{1j}(X_{a_{j-1}:n}-M_2)\right]\end{aligned}$$

$$C_2=(a_1-1)\beta_2(X_{a_1:n}-M_2)^2-(n-a_s)(X_{a_s:n}-M_2)^2-\sum_{j=1}^{s}(X_{a_j:n}-M_2)^2.$$

Third, making use of the approximate expressions in (2.5), (2.7), (2.8), (2.9), (2.10), and (2.11), we may approximate the likelihood equation of (2.3) as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq -\frac{1}{\sigma}\Big[2s+(a_1-1)(\alpha_1+\beta_1 Z_{a_1:n})-(n-a_s)Z_{a_s:n}^2-\sum_{j=1}^{s} Z_{a_j:n}^2 \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)\{(\alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n})Z_{a_j:n} \\ &\quad -(\alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n})Z_{a_{j-1}:n}\}\Big] \\ &=0.\end{aligned} \tag{2.19}$$

Upon solving the equations (2.19) and (2.13) for σ and θ, we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_3=\frac{-B_3+\sqrt{B_3^2-4A_3C_3}}{2A_3} \tag{2.20}$$

and

$$\hat{\theta}_3 = M_1\hat{\sigma}_3+M_2, \tag{2.21}$$

where

$$A_3 = 2s + (a_1 - 1)(\alpha_1 - \beta_1 M_1) - (n - a_s + s)M_1^2 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left[(\beta_{4j} + \gamma_{4j})M_1^2 - \alpha_{4j} M_1\right]$$

$$B_3 = (a_1 - 1)\beta_1 (X_{a_1:n} - M_2) + 2(n - a_s)M_1 (X_{a_s:n} - M_2) + 2M_1 \sum_{j=1}^{s}(X_{a_j:n} - M_2) + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left[(\alpha_{2j} - 2\beta_{2j} M_1 - \gamma_{2j} M_1 + \beta_{3j} M_1)(X_{a_j:n} - M_2) - (\alpha_{3j} - \beta_{3j} M_1 + \gamma_{2j} M_1 - 2\gamma_{3j} M_1)(X_{a_{j-1}:n} - M_2)\right]$$

$$C_3 = -(n - a_s)(X_{a_s:n} - M_2)^2 - \sum_{j=1}^{s}(X_{a_j:n} - M_2)^2 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left[\beta_{2j}(X_{a_j:n} - M_2)^2 + (\gamma_{2j} - \beta_{3j})(X_{a_j:n} - M_2)(X_{a_{j-1}:n} - M_2) - \gamma_{3j}(X_{a_{j-1}:n} - M_2)^2\right].$$

Fourth, making use of the approximate expressions in (2.7), (2.8), (2.9), (2.10), and (2.11), we may approximate the likelihood equation of (2.3) as follows;

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\Big[2s + (a_1 - 1)(\alpha_2 + \beta_2 Z_{a_1:n})Z_{a_1:n} - (n - a_s)Z^2_{a_s:n} - \sum_{j=1}^{s} Z^2_{a_j:n} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\big\{(\alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n}) Z_{a_j:n} - (\alpha_{3j} + \beta_{3j} Z_{a_j:n} + \gamma_{3j} Z_{a_{j-1}:n}) Z_{a_{j-1}:n}\big\}\Big] = 0. \tag{2.22}$$

Upon solving the equations (2.22) and (2.13) for σ and θ, we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_4 = \frac{-B_4 + \sqrt{B_4^2 - 4A_4 C_4}}{2A_4} \tag{2.23}$$

and

$$\hat{\theta}_4 = M_1 \hat{\sigma}_4 + M_2, \tag{2.24}$$

where

$$A_4 = 2s - (a_1 - 1)M_1(\alpha_2 - \beta_2 M_1) - (n - a_s + s)M_1^2 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left[(\beta_{4j} + \gamma_{4j})M_1^2 - \alpha_{4j} M_1\right]$$

$$B_4 = (a_1 - 1)(\alpha_2 - 2\beta_2 M_1)(X_{a_1:n} - M_2) + 2(n - a_s)M_1(X_{a_s:n} - M_2) + 2M_1\sum_{j=1}^{s}(X_{a_j:n} - M_2) + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left[(\alpha_{2j} - 2\beta_{2j} M_1 - \gamma_{2j} M_1 + \beta_{3j} M_1) \times (X_{a_j:n} - M_2) - (\alpha_{3j} - \beta_{3j} M_1 + \gamma_{2j} M_1 - 2\gamma_{3j} M_1)(X_{a_{j-1}:n} - M_2)\right]$$

$$C_4 = (a_1-1)\beta_2(X_{a_1:n}-M_2)^2-(n-a_s)(X_{a_s:n}-M_2)^2-\sum_{j=1}^{s}(X_{a_j:n}-M_2)^2$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\beta_{2j}(X_{a_j:n}-M_2)^2+(\gamma_{2j}-\beta_{3j})(X_{a_j:n}-M_2)(X_{a_{j-1}:n}-M_2)\right.$$
$$\left.-\gamma_{3j}(X_{a_{j-1}:n}-M_2)^2\right].$$

It's difficult to find the moments of all proposed estimators. So, we simulate the MSEs of all proposed estimators through Monte Carlo simulation method. The simulation procedure is repeated 10,000 times for the sample size $n=20(10)50$ and various choices of censoring ($m=n-s$ is the number of unobserved or missing data) under multiply Type-II censored samples. These values are given in Table 1.

From Table 1, the estimator $\hat{\sigma}_4$ is more efficient than the other estimators of the scale parameter σ in the sense of the MSE, and $\hat{\sigma}_2$ is generally more efficient than the estimators $\hat{\sigma}_1$ and $\hat{\sigma}_3$. The estimator $\hat{\theta}_4$ that use the estimator $\hat{\sigma}_4$ is more efficient than the other estimators of the location parameter θ in the sense of the MSE, and $\hat{\theta}_2$ that use the estimator $\hat{\sigma}_2$ is generally more efficient than the estimators $\hat{\theta}_1$ and $\hat{\theta}_3$. So we can recommend the proposed estimators $\hat{\sigma}_4$ and $\hat{\theta}_4$ of the scale and location parameters in the two-parameter Rayleigh distribution.

As expected, the MSE of all estimators decreases as sample size n increases. For fixed sample size, the MSE generally increases as m increases.

<Table 1> The relative MSEs for the location parameter θ and the scale parameter σ

n	m	a_j				
			$\hat{\theta}_1$	$\hat{\sigma}_1$	$\hat{\theta}_2$	$\hat{\sigma}_2$
20	0	1~20	0.042813	0.028502	0.042813	0.028502
	2	1~18	0.044991	0.032998	0.044991	0.032998
		3~20	0.048540	0.030774	0.047970	0.030442
		2~19	0.043407	0.030818	0.042632	0.030302
	4	2~17	0.046007	0.036191	0.045115	0.035532
		4~19	0.059168	0.036932	0.058652	0.036638
		3~18	0.052017	0.036259	0.051373	0.035852
		2~4 7~14 16~20	0.042145	0.028474	0.041428	0.028018
	5	3~17	0.053998	0.039857	0.053301	0.039386
		4~18	0.061601	0.040200	0.061052	0.039875
		2~6 10~19	0.043465	0.030863	0.042694	0.030346
	6	4~17	0.064405	0.044489	0.063810	0.044115
		1 2 6~9 12~15 17~20	0.042021	0.028333	0.042021	0.028333
30	0	1~30	0.025794	0.018394	0.025794	0.018394
	2	1~28	0.026589	0.020273	0.026589	0.020273
		3~30	0.027509	0.019359	0.027223	0.019181
		2~29	0.025180	0.019011	0.024808	0.018746
	4	2~27	0.026092	0.021000	0.025681	0.020685
		4~29	0.031722	0.021758	0.031482	0.021612
		3~28	0.028534	0.021449	0.028224	0.021244
		2~4 7~14 16~30	0.024762	0.018178	0.024409	0.017933
	5	3~27	0.029171	0.022544	0.028843	0.022318
		4~28	0.032501	0.023110	0.032250	0.022951
		2~6 10~19 21~30	0.024832	0.018189	0.024475	0.017941
	6	4~27	0.033349	0.024380	0.033082	0.024207
		1 2 6~9 12~15 17~30	0.025357	0.018239	0.025357	0.018239

<Table 1> (continued)

n	m	a_j	$\hat{\theta}_1$	$\hat{\sigma}_1$	$\hat{\theta}_2$	$\hat{\sigma}_2$
40	0	1～40	0.017474	0.012746	0.017474	0.012746
	2	1～38	0.017915	0.013717	0.017915	0.013717
		3～40	0.018181	0.013202	0.018018	0.013096
		2～39	0.016897	0.013047	0.016682	0.012890
	4	2～37	0.017281	0.013964	0.017055	0.013792
		4～39	0.020317	0.014505	0.020183	0.014425
		3～38	0.018716	0.014248	0.018546	0.014133
		2～4 7～14 16～40	0.016657	0.012566	0.016451	0.012416
	5	3～37	0.018917	0.014717	0.018744	0.014598
		4～38	0.020668	0.015108	0.020531	0.015026
		2～6 10～19 21～40	0.016653	0.012573	0.016446	0.012422
	6	4～37	0.020917	0.015625	0.020778	0.015540
		1 2 6～9 12～15 17～40	0.017167	0.012629	0.017167	0.012629
50	0	1～50	0.013431	0.010137	0.013431	0.010137
	2	1～48	0.013705	0.010760	0.013705	0.010760
		3～50	0.013849	0.010370	0.013735	0.010291
		2～49	0.012862	0.010212	0.012713	0.010097
	4	2～47	0.013101	0.010805	0.012946	0.010680
		4～49	0.015231	0.011228	0.015136	0.011164
		3～48	0.014158	0.011017	0.014040	0.010931
		2～4 7～14 16～50	0.012703	0.009900	0.012560	0.009792
	5	3～47	0.014306	0.011338	0.014184	0.011248
		4～48	0.015395	0.011569	0.015298	0.011504
		2～6 10～19 21～50	0.012719	0.009908	0.012575	0.009799
	6	4～47	0.015571	0.011917	0.015472	0.011849
		1 2 6～9 12～15 17～50	0.013149	0.010040	0.013149	0.010040

<Table 1> (continued)

n	m	a_j	$\hat{\theta}_3$	$\hat{\sigma}_3$	$\hat{\theta}_4$	$\hat{\sigma}_4$
20	0	1~20	0.042813	0.028502	0.042813	0.028502
	2	1~18	0.044991	0.032998	0.044991	0.032998
		3~20	0.048540	0.030774	0.047970	0.030442
		2~19	0.043407	0.030818	0.042632	0.030302
	4	2~17	0.046007	0.036191	0.045115	0.035532
		4~19	0.059168	0.036932	0.058652	0.036638
		3~18	0.052017	0.036259	0.051373	0.035852
		2~4 7~14 16~20	0.039136	0.027184	0.038663	0.026956
	5	3~17	0.053998	0.039857	0.053301	0.039386
		4~18	0.061601	0.040200	0.061052	0.039875
		2~6 10~19	0.041409	0.029875	0.040798	0.029514
	6	4~17	0.064405	0.044489	0.063810	0.044115
		1 2 6~9 12~15 17~20	0.035126	0.024955	0.035126	0.024955
30	0	1~30	0.025794	0.018394	0.025794	0.018394
	2	1~28	0.026589	0.020273	0.026589	0.020273
		3~30	0.027509	0.019359	0.027223	0.019181
		2~29	0.025180	0.019011	0.024808	0.018746
	4	2~27	0.026092	0.021000	0.025681	0.020685
		4~29	0.031722	0.021758	0.031482	0.021612
		3~28	0.028534	0.021449	0.028224	0.021244
		2~4 7~14 16~30	0.023384	0.017616	0.023133	0.017478
	5	3~27	0.029171	0.022544	0.028843	0.022318
		4~28	0.032501	0.023110	0.032250	0.022951
		2~6 10~19 21~30	0.023385	0.017618	0.023122	0.017469
	6	4~27	0.033349	0.024380	0.033082	0.024207
		1 2 6~9 12~15 17~30	0.021831	0.016427	0.021831	0.016427

<Table 1> (continued)

n	m	a_j				
			$\hat{\theta}_3$	$\hat{\sigma}_3$	$\hat{\theta}_4$	$\hat{\sigma}_4$
40	0	1~40	0.017474	0.012746	0.017474	0.012746
	2	1~38	0.017915	0.013717	0.017915	0.013717
		3~40	0.018181	0.013202	0.018018	0.013096
		2~39	0.016897	0.013047	0.016682	0.012890
	4	2~37	0.017281	0.013964	0.017055	0.013792
		4~39	0.020317	0.014505	0.020183	0.014425
		3~38	0.018716	0.014248	0.018546	0.014133
		2~4 7~14 16~40	0.015763	0.012211	0.015612	0.012123
	5	3~37	0.018917	0.014717	0.018744	0.014598
		4~38	0.020668	0.015108	0.020531	0.015026
		2~6 10~19 21~40	0.015780	0.012200	0.015624	0.012107
	6	4~37	0.020917	0.015625	0.020778	0.015540
		1 2 6~9 12~15 17~40	0.015025	0.011458	0.015025	0.011458
50	0	1~50	0.013431	0.010137	0.013431	0.010137
	2	1~48	0.013705	0.010760	0.013705	0.010760
		3~50	0.013849	0.010370	0.013735	0.010291
		2~49	0.012862	0.010212	0.012713	0.010097
	4	2~47	0.013101	0.010805	0.012946	0.010680
		4~49	0.015231	0.011228	0.015136	0.011164
		3~48	0.014158	0.011017	0.014040	0.010931
		2~4 7~14 16~50	0.012104	0.009581	0.011996	0.009514
	5	3~47	0.014306	0.011338	0.014184	0.011248
		4~48	0.015395	0.011569	0.015298	0.011504
		2~6 10~19 21~50	0.012114	0.009649	0.012002	0.009577
	6	4~47	0.015571	0.011917	0.015472	0.011849
		1 2 6~9 12~15 17~50	0.011658	0.009122	0.011658	0.009122

References

1. Balakrishnan, N. (1989). Approximate MLE of the scale parameter of the Rayleigh distribution with censoring, *IEEE Transactions on Reliability*, 38, 355-357.
2. Cohen, A. C. (1991). *Truncated and Censored Samples : Theory and Applications*, Marcel Dekker, Inc.
3. Han, J. T. and Kang, S. B. (2006). Estimation for the Rayleigh distribution with known parameter under multiply Type-II censoring, *Journal of the Korean Data & Information Science Society*, 17, 933-943.
4. Kang, S. B. (2005). Estimation for the extreme value distribution based on multiple Type-II censored samples, *Journal of the Korean Data & Information Science Society*, 16, 629-638.
5. Kang, S. B. and Lee, S. K. (2005). AMLEs for the exponential distribution based on multiple Type-II censored samples, *The Korean Communications in Statistics*, 12, 603-613.
6. Kang, S. B. and Park, S. M. (2005). Estimation for the exponentiated exponential distribution based on multiple Type-II censored samples, *The Korean Communications in Statistics*, 12, 643-652.
7. Kong, F. and Fei, H. (1996). Limit theorems for the maximum likelihood estimate under general multiply Type-II censoring, *Annals of the Institute of Statistical Mathematics*, 48, 731-755.
8. Upadhyay, S. K., Singh, U., and Shastri, V. (1996). Estimation of exponential parameters under multiply Type-II censoring, *Communications in Statistics- Simulation and Computation*, 25, 801-815.