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T-sum of bell-shaped fuzzy intervals

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Abstract

The usual arithmetic operations on real numbers can be extended to arithmetical operations on fuzzy intervals by means of Zadeh's extension principle based on a t-norm T. A t-norm is called consistent with respect to a class of fuzzy intervals for some arithmetic operation, if this arithmetic operation is closed for this class. It is important to know which t-norms are consistent with a particular type of fuzzy intervals. Recently, Dombi and Győrbíró proved that addition is closed if the Dombi t-norm is used with two bell-shaped fuzzy intervals. A result proved by Mesiar on a strict t-norm based shape preserving additions of LR-fuzzy intervals with unbounded support is recalled. As applications, we define a broader class of bell-shaped fuzzy intervals. Then we study t-norms which are consistent with these particular types of fuzzy intervals. Dombi and Győrbíró's results are special cases of the results described in this paper.

Keywords : Fuzzy number, t-norm, t-norm-based addition, bell-shape fuzzy interval, shape preserving.

1. Introduction

In recent years, fuzzy arithmetic has grown in importance as an advanced tool in fuzzy optimization and control theory. The usual arithmetic operations on the reals can be extended to arithmetical operations on fuzzy intervals by means of Zadeh's extension principle [32]. This principle is based on a triangular norm T. Fuzzy arithmetic based on the sup-(t-norm) convolution, with the controllability of the increase of fuzziness, enables us to construct more flexible and adaptable mathematical models for several intelligent technologies based on the use of approximate reasoning and fuzzy logic. Hence, much effort is desirable in order to find exact and good approximate computational formulas for fuzzy arithmetic

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operations. Most of the results are restricted to the cases that have bounded support. The most frequently used fuzzy intervals are LR-fuzzy intervals where L and R are left and right shape functions, respectively. The simplest forms of fuzzy quantities are triangular fuzzy intervals. These forms have been studied extensively and used in many applications. Some results for fuzzy arithmetic operations and their applications can be found in [5-8, 10-20, 23-30]. An important feature of t-norm-based arithmetical operations is that it provides a mean of controlling the growth of uncertainty during calculations. Namely, shape preserving arithmetic operations of LR-fuzzy intervals allow control of the resulting spread. In practical computation, it is natural to require the preserving of the shape of fuzzy intervals during addition and multiplication. Recently, Hong [16] showed that T_W, the weakest t\$-norm, is the only t-norm T that induces shape preserving multiplications of LR-fuzzy intervals. Mesiar [29] was interested, for given shapes, L and R, in which t-norm T induces a shape preserving addition of LR-fuzzy numbers. Recently, Dombi and Győrbíró [3] investigated the shape preserving additions of two bell-shaped membership functions. These representations and operations appear in many natural processes and are used in fuzzy neural networks [1, 9] for representing input and output values and measuring the error in the delta learning rule [33]. They proved that the addition operation with the Dombi operator is closed even in the case of infinite summation. These membership functions have unbounded supports.

In this paper, we recall a result [6, 29, 30] on the shape preserving addition of LR-fuzzy numbers with unbounded supports. As applications, we consider the T-sum of a broader class of bell-shaped fuzzy intervals where T is a strict t-norm. We study some specific T-sums, e.g., the Dombi-sum, the Hamacher-sum, the product-sum, the Schweizer and Sklar-sum, and the Frank-sum as examples. The results of Dombi and Győrbíró [3] are special cases of a result of this paper.

2. Basic definitions

A function, denoted L or R, is a shape function of fuzzy numbers iff $L(0)=1$ and L is a strictly decreasing continuous function on $[0,\infty)$. For instance, $L(x)=\max(0,1-x^p)$, $p>0$(which has bounded support and can be normalized as $L(1)=0$) ; $L(x)=e^{-x^p}$, $p>0$ or $L(x)=1/(1+x^p)$, $p>0$ (which has an unbounded support). Following Dubois and Prade [5], a fuzzy interval A is a so called LR-fuzzy interval, $A=(a,b,\alpha,\beta)_{LR}$, if the corresponding membership function satisfies for all $x\in R$

$$A(x)=\begin{cases}1 & \text{if } a\le x\le b\\ L\left(\frac{a-x}{\alpha}\right) & \text{if } x\le a,\ \alpha>0\\ R\left(\frac{x-b}{\beta}\right) & \text{if } b\le x,\ \beta>0\end{cases}$$

where $[a,b]$ is the peak of A, $\alpha>0$ and $\beta>0$ is the left and the right spread, respectively. If $R(x)=L(x)=1-x$, we denote $(a,b,\alpha,\beta)_{LR}$ by (a,b,α,β).

Recall that a triangular norm (briefly t-norm) is a binary operation T on the unit interval $[0,1]$ which is commutative, associative, and monotone and has 1 as a neutral element [21,22].

A continuous t-norm T strictly increasing on the open square $(0,1)^2$ is called a strict t-norm. Each strict t-norm is generated by an unbounded additive generator $f:[0,1]\to[0,\infty]$ with $f(1)=0$, where f is a continuous strictly decreasing bijection such that

$$T(x,y)=f^{[-1]}(f(x)+f(y)).$$

Let T be a given t-norm and let A_i, $i=1,2,\cdots,n$ be fuzzy intervals. Then their T-sum $C=A_1\oplus\cdots\oplus A_n$ is defined by the generalized extension principle of Zadeh [32] as

$$C(z)=\sup_{x_1+\cdots+x_n=z} T(A_1(x_1),\cdots,A_n(x_n)),\ z\in R.$$

If T is a strict t-norm with the additive generator f then

$$C(z)=f^{-1}\left(\inf_{x_1+\cdots+x_n=z} T(A_1(x_1),\cdots,A_n(x_n))\right)$$

3. The T-sum of bell-shaped fuzzy intervals

Recently, Dombi and Győrbíró [3] proved that additions are closed if the Dombi t-norm is used with two bell-shaped fuzzy intervals. In the following section we define a broader class of bell-shaped fuzzy intervals and study t-norms which are consistent with these particular types of fuzzy intervals as applications of shape-preserving properties. The results of Dombi and Győrbíró are special cases of the results of the work described in this paper.

We recall first the result of Mesier [6, 29, 30] on strict t-norm based shape preserving additions of the LR-fuzzy intervals with unbounded support.

Theorem 1. Let both L and R be shape functions with unbounded support. Let T be a strict t-norm with additive generator $f=(R^{-1})^p$ for some $p \geq 1$ and let $L(x)=R(x^k)$ for some $k>0$. Then the addition $\oplus_T$ based on T preserves the LR-shape of LR-fuzzy intervals and for $p \in (1, \infty)$

$$(a_1, b_1, \alpha_1, \beta_1)_{LR} \oplus_T (a_2, b_2, \alpha_2, \beta_2)_{LR} = (a_1+a_2, b_1+b_2, (\alpha_1^s+\alpha_2^s)^{1/s}, (\beta_1^q+\beta_2^q)^{1/q})_{LR}$$

where $1/p+1/q=1$, *i.e.* $q=p/(p-1)$ and $1/(pk)+1/s=1$, *i.e.* $s=pk/(pk-1)$.

We now define a broader class of bell-shaped fuzzy intervals.

Definition 1. Let f be the additive generator of a strict t-norm T. If $f \circ B^f_{c,d}(x)=((x-c)/d)^2$, then we call $B^f_{c,d}(x)=f^{-1}(((x-c)/d)^2)$ the bell-shaped membership function with respect to T, where c is the center and d is the width of the function.

Example 1. If $f(x)=((1-x)/x)^p$, $p>0$ is the additive generator of the *Dombi* $t-norm$ D where

$$D_p(x,y)=\frac{1}{1+\left(\left(\frac{1}{x}-1\right)^p+\left(\frac{1}{y}-1\right)^p\right)^{\frac{1}{p}}}$$

then $B^f_{c,d}(x)=1/(1+((x-c)/d)^{2/p})$.

Example 2. If $f(x)=\ln((p+(1-p)x)/x)$, $p \geq 0$ is the additive generator of the *Hamacher* $t-norm$ H where

$$H_p(x,y)=\frac{xy}{p+(1-p)(x+y-xy)}$$

then $B^f_{c,d}(x) = p/(e^{((x-c)/d)^2} + p - 1)$.

Example 3. If $f(x) = (1/p)(x^{-p} - 1)$, $p > 0$ is the additive generator of the *Schweizer* and *Sklar* $t-norm$ S where

$$S_p(x,y) = (x^{-p} + y^{-p} - 1)^{-\frac{1}{p}}$$

then $B^f_{c,d}(x) = (1/(p((x-c)/d)^2 + 1))^{1/p}$.

Example 4. If $f(x) = \log_p((p-1)/(p^x - 1))$, $p > 0$, $p \neq 1$ is the additive generator of the *Frank* $t-norm$ F where

$$F_p(x,y) = \log_p\left[1 + \frac{(p^x - 1)(p^y - 1)}{p - 1}\right]$$

then $B^f_{c,d}(x) = \log_p((p^{((x-c)/d)^2} + p - 1)/p^{((x-c)/d)^2})$.

Note. It is noted that if $f(x)$ is the additive generator of the *Dombi* $t-norm$, with $p = 1$ then $B^f_{c,d}(x) = 1/(1 + ((x-c)/d)^2)$ is the definition of one bell-shape membership function defined by Dombi and Győrbíró ([3] Definition 6.). If $f(x)$ is the additive generator of the *Hamacher* $t-norm$ with $p = 1$ then $B^f_{c,d}(x) = e^{-((x-c)/d)^2}$ is the definition of another bell-shape membership function defined by Dombi and Győrbíró. ([3] Definition 11.)

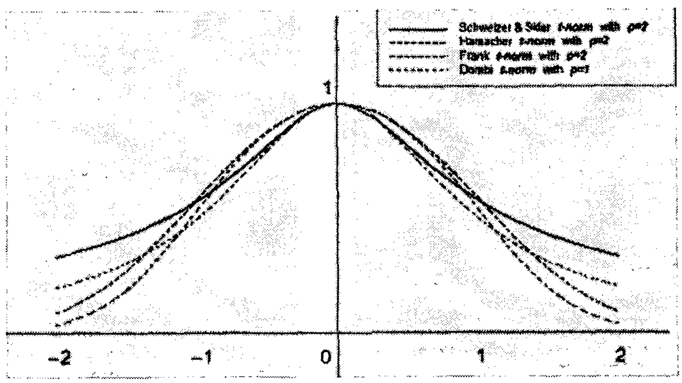

Fig 1. Various types of bell-shaped fuzzy intervals

As a direct application of Theorem 1, we have the following shape-preserving addition of bell-shaped fuzzy intervals.

Theorem 2. Let T be a strict t-norm with additive generator, f, and let $B^f_{c_i,d_i}(x)$, $i=1,2,\cdots$ be a sequence of the bell-shaped membership functions. Then the addition $\oplus_T$ based on T preserves the bell-shape of fuzzy intervals and

$$B^f_{c_1,d_1}\oplus_T\cdots\oplus_T B^f_{c_n,d_n}=B^f_{c_1+\cdots+c_n,(d_1^2+\cdots+d_n^2)^{1/2}}$$

Additionally, if $\sum_{i=1}^{\infty}c_i=C$ and $\sum_{i=1}^{\infty}d_i^2=D$, then

$$\lim_{n\to\infty}B^f_{c_1,d_1}\oplus_T\cdots\oplus_T B^f_{c_n,d_n}=B^f_{C,D^{1/2}}.$$

Proof. It is enough to show that for $n=2$. By the definition of the bell-shaped fuzzy intervals, $f\circ B^f_{0,1}(x)=x^2$, and hence $f=(B^{f\,-1}_{0,1})^2$. It is also easy to check that $B^f_{c,d}=(c,c,d,d)_{LR}$ with $L=R=B^f_{0,1}$. Then, as mentioned in Section 2, the investigated membership function is

$$\begin{aligned}B^f_{c_1,d_1}\oplus_T B^f_{c_2,d_2}(z)&=f^{-1}\left(\inf_{x_1+x_2=z}f\left(B^f_{c_1,d_1}(x_1)\right)+f\left(B^f_{c_2,d_2}(x_2)\right)\right)\\&=\left(c_1,c_1,d_1,d_1\right)_{LR}\oplus_T\left(c_2,c_2,d_2,d_2\right)_{LR}\\&=\left(c_1+c_2,c_1+c_2,\left(d_1^2+d_2^2\right)^{1/2},\left(d_1^2+d_2^2\right)^{1/2}\right)_{LR}\end{aligned}$$

where the last equality comes from Theorem 1. Therefore we have

$$B^f_{c_1,d_1}\oplus_T B^f_{c_2,d_2}(z)=B^f_{c_1+c_2,(d_1^2+d_2^2)^{1/2}}(z),$$

and hence we have

$$\begin{aligned}\lim_{n\to\infty} B^f_{c_1,d_1} \oplus_T \cdots \oplus_T B^f_{c_n,d_n}(z) &= \lim_{n\to\infty} B^f_{c_1+\cdots+c_n,(d_1^2+\cdots+d_n^2)^{1/2}}(z) \\ &= \lim_{n\to\infty} f^{-1}\left(\left(\frac{z-(c_1+\cdots+c_n)}{(d_1^2+\cdots+d_n^2)^{1/2}}\right)^2\right) \\ &= \lim_{n\to\infty} f^{-1}\left(\left(\frac{z-C}{D^{1/2}}\right)^2\right) \\ &= B^f_{C,D^{1/2}}(z)\end{aligned}$$

which completes the proof.

Corollary 1. Let T be a strict t-norm with additive generator, f, and let $B^f_{c_i,d_i}(x)$, $i=1,2,\cdots$ be the bell-shape membership functions, all having the same width. We have

$$B^f_{c_1,d} \oplus_T \cdots \oplus_T B^f_{c_n,d} = B^f_{c_1+\cdots+c_n,\sqrt{n}\,d}$$

Note 1. We see that Theorem 7 [3] is a special case of Corollary 1.

Corollary 2. Let T be a strict t-norm with additive generator, f, and let $B^f_{c_i,d_i}(x)$, $i=0,1,\cdots$ an infinite series of the bell-shape membership functions with $d_i = dq^i$, $q<1$ and $d>0$. If the series $\sum_{i=0}^{\infty} c_i = C$ is convergent, then we have

$$\lim_{n\to\infty} B^f_{c_1,d_1} \oplus_T \cdots \oplus_T B^f_{c_n,d_n} = B^f_{C,d/\sqrt{1-q^2}}.$$

Note 2. We see that Theorem 8 [3] and Corollary 10 [3] are special cases of Corollary 2. For this, we see the following examples.

Example 5. Let $B^f_{c,d}(x) = 1/(1+((x-c)/d)^{2/p})$ where $f(x)=((1-x)/x)^p$, $p>0$ is the additive generator of the *Dombi* $t-norm$. Then, by Theorem 5,

$$\sup_{x_1+\cdots+x_n=z} \frac{1}{1+\sum_{i=1}^{n}((x_i-c_i)/d)^{2/p}} = B^f_{c_1,d} \oplus_T \cdots \oplus_T B^f_{c_n,d}(z)$$

$$= B^f_{\sum_{i=1}^{n} c_i, d\sqrt{n}}(z)$$

$$= \frac{1}{1+\left(\left(z-\sum_{i=1}^{n} c_i\right)/d\sqrt{n}\right)^{2/p}} \quad \text{(Theorem 2 [3])}$$

Also, we have for $q<1$ and $\sum_{i=0}^{\infty} c_i = C$

$$\sup_{x_0+\cdots+x_n=z} \frac{1}{1+\sum_{i=0}^{n}((x_i-c_i)/dq^i)^{2/p}} = B^f_{c_0,d} \oplus_T \cdots \oplus_T B^f_{c_n,dq^n}(z)$$

$$= B^f_{\sum_{i=0}^{n} c_i, d\sqrt{\sum_{i=1}^{n} q^{2i}}}(z)$$

$$= \frac{1}{1+\left(\left(z-\sum_{i=0}^{n} c_i\right)/d\sqrt{\sum_{i=1}^{n} q^{2i}}\right)^{2/p}} \quad \text{(Lemma 9}$$

[3])

and hence , noting that $\lim_{n\to\infty} \sum_{i=0}^{n} q^{2i} = 1/(1-q^2)$,

$$\lim_{n\to\infty} \sup_{x_0+\cdots+x_n=z} \frac{1}{1+\sum_{i=0}^{n}((x_i-c_i)/dq^i)^{2/p}}$$

$$= \frac{1}{1+\left((z-C)/\left(d/\sqrt{1-q^2}\right)\right)^{2/p}}, \quad \text{(Theorem8 [3])}$$

Example 6. Let $B^f_{c,d}(x) = p/(e^{((x-c)/d)^2}+p-1)$ where $f(x)=\ln((p+(1-p)x)/x)$, $p \geq 0$ is the additive generator of the *Hamacher* $t-norm$. Then, by Theorem 2,

$$\sup_{x_1+\cdots+x_n=z} \frac{\prod_{i=1}^{n} p/\left(e^{((x-c_i)/d)^2}+p-1\right)}{p+(1-p)\left(\sum_{i=1}^{n} p/\left(e^{((x-c_i)/d)^2}+p-1\right)-\prod_{i=1}^{n} p/\left(e^{((x-c_i)/d)^2}+p-1\right)\right)}$$

$$= B^f_{c_1,d} \oplus_T \cdots \oplus_T B^f_{c_n,d}(z)$$

$$= B^f_{\sum_{i=1}^{n} c_i,\, d\sqrt{n}}(z)$$

$$= p/\left(e^{-\left(\left(z-\sum_{i=1}^{n} c_i\right)/d\sqrt{n}\right)^2}+p-1\right)$$

Also, we have for $q<1$ and $\sum_{i=0}^{\infty} c_i = C$

$$\sup_{x_0+\cdots+x_n=z} \frac{\prod_{i=0}^{n} p/\left(e^{((x-c_i)/dq^i)^2}+p-1\right)}{p+(1-p)\left(\sum_{i=1}^{n} p/\left(e^{((x-c_i)/dq^i)^2}+p-1\right)-\prod_{i=1}^{n} p/\left(e^{((x-c_i)/dq^i)^2}+p-1\right)\right)}$$

$$= B^f_{c_0,d} \oplus_T \cdots \oplus_T B^f_{c_n,dq^n}(z)$$

$$= B^f_{\sum_{i=0}^{n} c_i,\, d\sqrt{\sum_{i=1}^{n} q^{2i}}}(z)$$

$$= p/\left(e^{-\left(\left(z-\sum_{i=1}^{n} c_i\right)/d\sqrt{\sum_{i=1}^{n} q^{2i}}\right)^2}+p-1\right)$$

Hence, noting that $\lim_{n\to\infty} \sum_{i=0}^{n} q^{2i} = 1/(1-q^2)$,

$$\lim_{n\to\infty} \sup_{x_0+\cdots+x_n=z} \frac{\prod_{i=0}^{n} p/\left(e^{((x-c_i)/dq^i)^2}+p-1\right)}{p+(1-p)\left(\sum_{i=1}^{n} p/\left(e^{((x-c_i)/dq^i)^2}+p-1\right)-\prod_{i=1}^{n} p/\left(e^{((x-c_i)/dq^i)^2}+p-1\right)\right)}$$

$$= p/(e^{-((z-C)/d\sqrt{1/(1-q^2)})^2}+p-1)$$

$$\lim_{n\to\infty} \sup_{x_0+\cdots+x_n=z} \prod_{i=0}^{n} e^{-((x_i-c_i)/dq^i)^2}$$

$$= e^{-((z-C)/d\sqrt{1/(1-q)^2})^2}$$

If $p=1(product\ t-norm)$, then

$$\sup_{x_1+\cdots+x_n=z}\prod_{i=1}^{n}e^{-((x_i-c_i)/d)^2}=e^{-\left(\left(z-\sum_{i=1}^{n}c_i\right)/d\sqrt{n}\right)^2}$$

$$\begin{aligned}\sup_{x_0+\cdots+x_n=z}\prod_{i=0}^{n}e^{-((x_i-c_i)/dq_i)^2}&=B^f_{c_0,d}\oplus_T\cdots\oplus_T B^f_{c_n,dq^n}(z)\\&=B^f_{\sum_{i=0}^{n}c_i,d\sqrt{\sum_{i=1}^{n}q^{2i}}}(z)\\&=e^{-\left(\left(z-\sum_{i=1}^{n}c_i\right)/d\sqrt{\sum_{i=1}^{n}q^{2i}}\right)^2}\end{aligned}$$

and

$$\lim_{n\to\infty}\sup_{x_0+\cdots+x_n=z}\prod_{i=0}^{n}e^{-((x_i-c_i)/dq^i)^2}$$
$$=e^{-((z-C)/d\sqrt{1/(1-q)^2})^2}$$

Example 7. Let $B^f_{c,d}(x)=\left(p/\left(e^{((x-c)/d)^2}+p-1\right)\right)^{1/p}$, where $f(x)=(1/p)(x^{-p}-1)$, $p\geq 0$ is the additive generator of *Schweizer* and the *Sklar t−norm*. Then, by Theorem 2,

$$\begin{aligned}\sup_{x_1+\cdots+x_n=z}\sum_{i=1}^{n}\left(1/\left(p((x_i-c_i)/d)^2+1\right)^{-1}-1\right)^{-\frac{1}{p}}&=B^f_{c_1,d}\oplus_T\cdots\oplus_T B^f_{c_n,d}(z)\\&=B^f_{\sum_{i=1}^{n}c_i,d\sqrt{n}}(z)\\&=\left(1/\left(p\left(\left(x-\sum_{i=1}^{n}c_i\right)/d\sqrt{n}\right)^2+1\right)\right)^{1/p}\end{aligned}$$

Also, we have for $q<1$ and $\sum_{i=0}^{\infty}c_i=C$

$$\begin{aligned}\sup_{x_0+\cdots+x_n=z}\sum_{i=0}^{n}\left(1/\left(p\left((x_i-c_i)/dq^i\right)^2+1\right)^{-1}-1\right)^{-\frac{1}{p}}&=B^f_{c_0,d}\oplus_T\cdots\oplus_T B^f_{c_n,dq^n}(z)\\&=B^f_{\sum_{i=0}^{n}c_i,d\sqrt{\sum_{i=1}^{n}q^{2i}}}(z)\\&=\left(1/\left(p\left(\left(x-\sum_{i=1}^{n}c_i\right)/d\sqrt{\sum_{i=1}^{n}q^{2i}}\right)^2+1\right)\right)^{1/p}\end{aligned}$$

and hence , noting that $\lim_{n\to\infty} \sum_{i=0}^{n} q^{2i} = 1/(1-q^2)$,

$$\lim_{n\to\infty} \sup_{x_0+\cdots+x_n=z} \sum_{i=0}^{n} \left(1/\left(p\left((x_i-c_i)/dq^i\right)^2+1\right)^{-1}-1\right)^{-\frac{1}{p}}$$
$$= \left(1/p\left((x-C)/d\sqrt{1/(1-q^2)}\right)^2+1\right)^{1/p}$$

Example 8. Let $B^f_{c,d}(x) = \log_p((p^{((x-c)/d)^2}+p-1)/p^{((x-c)/d)^2})$ where $f(x) = \log_p((p-1)/(p^x-1))$, $p>0$ is the additive generator of the *Frank t-norm*. Then, by Theorem 2,

$$\sup_{x_1+\cdots+x_n=z} \log_p\left[1+\frac{\prod_{i=1}^{n}\left(p^{\log_p\left(\left(p^{((x-c_i)/d)^2}+p-1\right)/p^{((x-c_i)/d)^2}\right)}-1\right)}{p-1}\right]$$
$$= B^f_{c_1,d} \oplus_T \cdots \oplus_T B^f_{c_n,d}(z)$$
$$= B^f_{\sum_{i=1}^{n} c_i, d\sqrt{n}}(z)$$
$$= \log_p\left(\left(p^{\left(\left(x-\sum_{i=1}^{n} c_i\right)/d\sqrt{n}\right)^2}+p-1\right)\Big/p^{\left(\left(x-\sum_{i=1}^{n} c_i\right)/d\sqrt{n}\right)^2}\right)$$

Also, we have for $q<1$ and $\sum_{i=1}^{\infty} c_i = C$

$$\sup_{x_0+\cdots+x_n=z} \log_p\left[1+\frac{\prod_{i=0}^{n}\left(p^{\log_p\left(\left(p^{((x-c_i)/dq^i)^2}+p-1\right)/p^{((x-c_i)/dq^i)^2}\right)}-1\right)}{p-1}\right]$$

$$= B^f_{c_0,d} \oplus_T \cdots \oplus_T B^f_{c_n,dq^n}(z)$$
$$= B^f_{\sum_{i=0}^{n} c_i, d\sqrt{\sum_{i=0}^{n} q^{2i}}}(z)$$
$$= \log_p\left(\left(p^{\left(\left(x-\sum_{i=0}^{n} c_i\right)/d\sqrt{\sum_{i=0}^{n} q^{2i}}\right)^2}+p-1\right)\Big/p^{\left(\left(x-\sum_{i=1}^{n} c_i\right)/d\sqrt{\sum_{i=0}^{n} q^{2i}}\right)^2}\right)$$

and hence , noting that $\lim_{n\to\infty} \sum_{i=0}^{n} q^{2i} = 1/(1-q^2)$,

$$\lim_{n\to\infty} \sup_{x_0+\cdots+x_n=z} \log_p \left[1 + \frac{\prod_{i=0}^{n} \left(p^{\log_p\left(\left(p^{((x-c_i)/dq^i)^2} + p - 1\right)/p^{((x-c_i)/dq^i)^2}\right)} - 1 \right)}{p-1} \right]$$
$$= \log_p\left(\left(p^{((x-C)/d\sqrt{1/(1-q^2)})^2} + p - 1\right)/p^{((x-C)/d\sqrt{1/(1-q^2)})^2}\right)$$

4. Conclusion

In this paper, we recalled a result on the shape preserving addition of LR-fuzzy numbers with unbounded supports. As applications, we defined a broader class of bell-shaped fuzzy intervals and presented sums of bell-shaped fuzzy intervals in the sup-T sum construction. Results of sums of bell-shaped fuzzy intervals of Dombi and Győrbíró are special cases of the results presented in this paper. Results of sums of sigmoid-shaped fuzzy intervals of Dombi and Győrbíró can be generalized in a similar manner.

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