점성감쇠 모델을 위한 새로운 동적 압축 방법
Alternative Dynamic Condensation Methods for Viscously Damped Models

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ABSTRACT

Two ways can be used for dynamic condensation of viscously damped structural models. One is reducing the model in physical space at first and then transferring it to state space. The other is condensing the model in state space directly. Two iterative schemes for each way are given respectively. Hence four iterative schemes for dynamic condensation of nonclassically damped models are discussed in this paper. A high building with a tuned mass damper is applied to show the efficiency of these schemes.

Keywords: *Dynamic condensation, iterative scheme, master degree of freedom, slave degree of freedom*

1. Introduction

The dynamic condensation was firstly proposed by Guyan (1965) and Irons (1965). Since the dynamic effects were ignored in this method, it is a static condensation and only exact for static problems. Hence, many schemes have been proposed to improve the accuracy. Among them, the most accurate methods are the iterative schemes (Miller, 1980 O'Callahan, 1989 Suarez et al., 1992 and Qu et al., 1998) because the dynamic condensation matrix is updated repeatedly until the desired convergent values are obtained.

An iterative method for dynamic condensation of viscously damped systems was proposed by Qu (1998). In this method, two governing equations for the dynamic condensation matrix, which relates the eigenpaires associated to the master and slave degrees of freedom in state space, were derived. Since the eigenvectors and eigenvalues of the reduced model do not include in the equation, it is unnecessary to solve for the eigenproblem of the reduced model for all the iterations.

In this paper, the dynamic condensation of viscously damped structural models both in physical space and state space are presented and discussed. Two iterative schemes for each way are proposed respectively. The effects of dynamic condensation space, condensation schemes, and the damping of the model on the accuracy of the damping ratios and frequencies are discussed by using a high building with tuned mass damper.

2. Condensation in Physical Space

The dynamic equations of an n-dimensional viscously damped system can be written in a matrix form as

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$$M\ddot{X}(t) + C\dot{X}(t) + KX(t) = f(t) \tag{1}$$

in which matrix M , C , and K are assumed to be positive definite, positive semi-definite and positive semi-definite, respectively. $f(t)$ is an external force vector. The eigenproblem of the corresponding undamped system is

$$K\Phi = M\Phi\Lambda \tag{2}$$

Λ represents a diagonal matrix with the system eigenvalues and the columns of Φ are the corresponding eigenvectors. Here the eigenvalues in matrix Λ are arranged in an ascending form,

$$0 < \lambda_1 \le \lambda_2 \le \cdots\cdots \le \lambda_n \tag{3}$$

If only the formal m eigenpairs are considered here. Eq. (2) yields

$$K\Phi_m = M\Phi_m\Lambda_{mm} \tag{4}$$

Assume that the total degrees of freedom of the full model (n) are divided into the masters degrees of freedom (m), which will be retained in the reduced model, and the slaves degrees of freedom (s), which will be deleted from the full model. According to this division, Eqs. (1) and (4) can be rewritten in a partitioned form as

$$\begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix}\begin{bmatrix} \ddot{X}_m \\ \ddot{X}_s \end{bmatrix} + \begin{bmatrix} C_{mm} & C_{ms} \\ C_{sm} & C_{ss} \end{bmatrix}\begin{bmatrix} \dot{X}_m \\ \dot{X}_s \end{bmatrix} + \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix}\begin{bmatrix} X_m \\ X_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix}\begin{bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{bmatrix} = \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix}\begin{bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{bmatrix}\Lambda_{mm} \tag{6}$$

There are two definitions of the dynamic condensation matrix R_P in physical space. (1) The dynamic condensation matrix is defined as the relationship of the displacements between the master and the slave degrees of freedom, that is

$$X_s = R_P X_m \tag{7}$$

(2) It indicates the relationships of the eigenvectors between the master degrees of freedom and the slave, i.e.

$$\Phi_{sm} = R_P \Phi_{mm} \tag{8}$$

According to the theory of modal superposition of the displacement, the displacements of the full model can be expressed approximately by using the former eigenvectors as

$$X = \Phi_m q \tag{9}$$

or in a partitioned form

$$X = \begin{bmatrix} X_m \\ X_s \end{bmatrix} = \begin{bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{bmatrix} q \tag{10}$$

Introducing Eq. (10) into Eq. (7) yields

$$\Phi_{sm} q = R_P \Phi_{mm} q \tag{11}$$

Since the vector q is a function of time t, one can get definition Eq. (8) from Eq. (11) easily.

When we have the dynamic condensation matrix, the system matrices of the reduced model are defined as Qu (1998)

$$M_R = M_{mm} + R_P^T M_{sm} + M_{ms} R_P + R_P^T M_{ss} R_P \tag{12a}$$

$$C_R = C_{mm} + R_P^T C_{sm} + C_{ms} R_P + R_P^T C_{ss} R_P \tag{12b}$$

$$K_R = K_{mm} + R_P^T K_{sm} + K_{ms} R_P + R_P^T K_{ss} R_P \tag{12c}$$

Then the eigenproblem of the reduced model is

$$A_{RP}\Psi_{mm} = B_{RP}\Psi_{mm}Q_{mm} \tag{13}$$

where Ψ_{mm} is the complex eigenvector matrix and Q_{mm} is the complex eigenvalue matrix. A , B are the system matrices in state space as

$$A_{RP} = \begin{bmatrix} K_R & 0 \\ 0 & -M_R \end{bmatrix} \qquad B_{RP} = \begin{bmatrix} -C_R & -M_R \\ -M_R & 0 \end{bmatrix}. \tag{14}$$

It can be seen from Eq. (12) that the reduced model is only dependent upon the dynamic condensation matrix when the full model is given. Hence, how to find a highly accurate dynamic condensation matrix is a main focus of most of the papers.

Two efficient iterative schemes have been proposed to obtain the dynamic condensation matrix R_P for the undamped system.

2.1 Scheme I

Expanding the under part of Eq. (6) yields

$$K_{sm}\Phi_{mm} + K_{ss}\Phi_{sm} = M_{sm}\Phi_{mm}\Lambda_{mm} + M_{ss}\Phi_{sm}\Lambda_{mm} \tag{15}$$

Moving $K_{sm}\Phi_{mm}$ from the left side of Eq. (15) to the right side and using Eq. (8), one has

$$K_{ss}R_P\Phi_{mm} = [M_{sm} + M_{ss}R_P]\Phi_{mm}\Lambda_{mm} - K_{sm}\Phi_{mm} \tag{16}$$

From Eq. (16), the governing equation of dynamic condensation matrix R_P may be derived as

$$R_P = K_{ss}^{-1}[(M_{sm} + M_{ss}R_P)\Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1} - K_{sm}] \tag{17}$$

Introducing the expression (Qu, 1998)

$$M_R^{-1}K_R = \Phi_{mm}\Lambda_{mm}\Phi_{mm}^{-1} \tag{18}$$

into Eq. (18) leads

$$R_P = K_{ss}^{-1}[(M_{sm} + M_{ss}R_P)M_R^{-1}K_R - K_{sm}] \tag{19}$$

The iterative forms for Eqs. (17) and (19) are

$$R_P^{(i+1)} = K_{ss}^{-1}[(M_{sm} + M_{ss}R_P^{(i+1)})\Phi_{mm}^{(i)}\Lambda_{mm}^{(i)}(\Phi_{mm}^{(i)})^{-1} - K_{sm}] \quad (i = 0.1,2,...) \tag{20}$$

$$R_P^{(i+1)} = K_{ss}^{-1}[(M_{sm} + M_{ss}R_P^{(i)})(M_R^{(i)})^{-1}K_R^{(i)} - K_{sm}] \quad (i = 0.1,2,...) \tag{21}$$

The initial approximation of the dynamic condensation matrix is

$$R_P^{(0)} = -K_{ss}^{-1}K_{sm} \tag{22}$$

2.2 Scheme II

According to the classical subspace iteration method (Bathe and Wilson, 1972), the $i+1$th subspace $Y_m^{(i+1)}$ is defined as

$$KY_m^{(i+1)} = M\Phi_m^{(i)} \tag{23}$$

where $\Phi_m^{(i)}$ is the ith approximation of the former m eigenvectors. The $i+1$th approximate eigenvector matrix can be expressed as

$$\Phi_m^{(i+1)} = Y_m^{(i+1)}Q_{mm}^{(i+1)} \tag{24}$$

in which $Q_{mm}^{(i+1)}$ is the eigenvector matrix of the reduced model defined by the subspace $Y_m^{(i+1)}$. Using the definition Eq. (8), one has

$$R_P^{(i+1)} = Y_{sm}^{(i+1)}(Y_{mm}^{(i+1)})^{-1} \tag{25}$$

Introducing Eq. (23) into Eq. (25) and considering the definition Eq. (8) again, we have

$$R_P^{(i+1)} = (D_{sm} + D_{ss}R_P^{(i)})(D_{mm} + D_{ms}R_P^{(i)})^{-1} \quad (i = 0.1,2,...) \tag{26}$$

where

$$D = K^{-1}M = \begin{bmatrix} D_{mm} & D_{ms} \\ D_{sm} & D_{ss} \end{bmatrix} \tag{27}$$

The initial approximation of the dynamic condensation matrix is

$$R_P^{(0)} = D_{sm}D_{mm}^{-1} \tag{28}$$

3. Condensation in State Space

The dynamic Eq. (1) can be rewritten in state space as

$$AZ(t) - BZ(t) = F(t) \tag{29}$$

where the system matrices A, B and vectors Z, F are defined as

$$A = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}, \quad B = \begin{bmatrix} -C & -M \\ -M & 0 \end{bmatrix} \tag{30}$$

$$Z(t) = \begin{Bmatrix} X(t) \\ \dot{X}(t) \end{Bmatrix}, \quad F(t) = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix} \tag{31}$$

Relatively, the eigenvalue problem for the lower m eigenpairs is

$$A\tilde{\Psi}_m = B\tilde{\Psi}_m\tilde{\Omega}_{mm} \tag{32}$$

The complex eigenvector matrix $\tilde{\Psi}_m$ and eigenvalue or spectral matrix $\tilde{\Omega}_{mm}$ have the forms

$$\tilde{\Psi}_m = \begin{bmatrix} \Psi_m & \Psi_m^* \\ \Psi_m\Omega_{mm} & \Psi_m^*\Omega_{mm}^* \end{bmatrix}, \quad \tilde{\Omega}_{mm} = \begin{bmatrix} \Omega_{mm} & 0 \\ 0 & \Omega_{mm}^* \end{bmatrix} \tag{33}$$

The superscript "*" denotes complex conjugation. Here the eigenvalues in matrix $\tilde{\Omega}_{mm}$ are an arranged in ascending form. The eigenvector matrix is assumed to have been normalized such that

$$\tilde{\Psi}_m^T A\tilde{\Psi}_m = \tilde{\Omega}_{mm}, \quad \tilde{\Psi}_m^T B\tilde{\Psi}_m = I_{mm} \tag{34}$$

where I is an unit matrix. According to the division of the degrees of freedom, Eq. (32) can be written in a partitioned form as

$$\begin{bmatrix} A_{mm} & A_{ms} \\ A_{sm} & A_{ss} \end{bmatrix}\begin{bmatrix} \tilde{\Psi}_{mm} \\ \tilde{\Psi}_{sm} \end{bmatrix} = \begin{bmatrix} B_{mm} & B_{ms} \\ B_{sm} & B_{ss} \end{bmatrix}\begin{bmatrix} \tilde{\Psi}_{mm} & \tilde{\Omega}_{mm} \\ \tilde{\Psi}_{sm} & \tilde{\Omega}_{mm} \end{bmatrix} \tag{35}$$

Similarly, there are two definitions for the dynamic condensation matrix in state space R_s, i.e.

$$Z_s = R_S Z_m \tag{36}$$

$$\tilde{\Psi}_{sm} = R_S\tilde{\Psi}_{mm} \tag{37}$$

It can also be proven that these two definitions are equivalent if only the former m eigenpairs are considered for the full model.

3.1 Scheme I

Expanding the under part of Eq. (35) yields

$$A_{sm}\tilde{\Psi}_{mm} + A_{ss}\tilde{\Psi}_{sm} = B_{sm}\tilde{\Psi}_{mm}\tilde{Q}_{mm} + B_{ss}\tilde{\Psi}_{sm}\tilde{Q}_{mm} \tag{38}$$

which can be rewritten as

$$\tilde{\Psi}_{sm} = A_{ss}^{-1}(B_{sm}\tilde{\Psi}_{mm}\tilde{Q}_{mm} + B_{ss}\tilde{\Psi}_{sm}\tilde{Q}_{mm} - A_{sm}\tilde{\Psi}_{mm}) \tag{39}$$

Substituting Eq. (37) into Eq. (39) results

$$R_s = A_{ss}^{-1}\left[(B_{sm} + B_{ss}R_s)\tilde{\Psi}_{mm}\tilde{Q}_{mm}\tilde{\Psi}_{mm}^{-1} - A_{sm}\right] \tag{40}$$

Introducing the expression (Qu, 1998)

$$B_R^{-1}A_R = \tilde{\Psi}_{mm}\tilde{Q}_{mm}\tilde{\Psi}_{mm}^{-1} \tag{41}$$

into Eq. (40) yields

$$R_S = A_{ss}^{-1}\left[(B_{sm} + B_{ss}R_s)B_R^{-1}A_R - A_{sm}\right] \tag{42}$$

The iterative forms of Eqs. (40) and (42) are

$$R_S^{(i+1)} = A_{ss}^{-1}\left[(B_{sm} + B_{ss}R_S^{(i)})\tilde{\Psi}_{mm}^{(i)}\tilde{Q}_{mm}^{(i)}(\tilde{\Psi}_{mm}^{(i)})^{-1} - A_{sm}\right](i = 0,1,2,\ldots) \tag{43}$$

$$R_S^{(i+1)} = A_{ss}^{-1}\left[(B_{sm} + B_{ss}R_S^{(i)})(B_R^{(i)})^{-1}A_R^{(i)} - A_{sm}\right] \quad (i = 0,1,2,\ldots) \tag{44}$$

The initial approximation of the dynamic condensation matrix is

$$R_S^{(0)} = A_{ss}^{-1}A_{sm} \tag{45}$$

3.2 Scheme II

According to the subspace iteration method for nonclassically damped system (Leung, 1995), the $i+1$ th approximate subspace is defined as

$$AY_m^{(i+1)} = B\tilde{\Psi}_m^{(i)} \tag{46}$$

where $\tilde{\Psi}_m^{(i)}$ is the i th approximation of the former m eigenvectors. The $i+1$ th approximate eigenvector matrix can be expressed as

$$\tilde{\Psi}_m^{(i+1)} = Y_m^{(i+1)}Q_{mm}^{(i+1)} \tag{47}$$

in which $Q_{mm}^{(i+1)}$ is the eigenvector matrix of the reduced model defined by the subspace $Y_m^{(i+1)}$. Substituting Eq. (47) into Eq. (37) and rearranging it yields

$$R_S^{(i+1)} = Y_{sm}^{(i+1)}(Y_{mm}^{(i+1)})^{-1} \tag{48}$$

Introducing Eq. (46) into Eq. (48) and considering the definition Eq. (37) again, one has

$$R_S^{(i+1)} = (G_{sm} + G_{ss}R_S^{(i)})(G_{mm} + G_{ms}R_S^{(i)})^{-1} \quad (i = 0,1,2,\ldots) \tag{49}$$

where

$$G = A^{-1}B = \begin{bmatrix} G_{mm} & G_{ms} \\ G_{sm} & G_{ss} \end{bmatrix} \tag{50}$$

The initial approximation of the dynamic condensation matrix is

$$R_S^{(0)} = G_{sm}G_{mm}^{-1} \tag{51}$$

4. Numerical Example

The schemes above have been tested on a 40-story tall building with Tuned Mass Damper (TMD). Each story unit of the building is identically constructed with a story height of 4 m, mass m_i = 1290 tons, stiffness $k_i = 10^6 kN/m$, and damping C_i = 14260 $kN \cdot s/m$ for i=1,2,⋯,40. The building is symmetric in both lateral directions and the mass center coincides with the elastic center, so that there is no coupled lateral-torsional motions. Only one direction motion will be considered. The mass of the damper is 258 tons, which is 20% of a floor mass. The stiffness and damping coefficient of the damper are $300.9 kN/m$ and $83.592 kN \cdot s/m$.

The former five damping frequencies and damping ratios for two cases are listed in Table 1. The degrees of freedom pertaining to the 10th, 20th, 30th, 40th floors and TMD are selected as the master degrees of freedom when dynamic condensation is applied to the tall building. Eight cases listed in Table 2 are considered here. The errors of the damping ratios and frequencies of the reduced model in the former ten iterations for the eight cases are drawn in Fig. 1.

Table 1. The former five damping ratios and frequencies

C=0.2Ci			C=0.04Ci		
No.	ξ	ω	No.	ξ	ω
1	.014364	1.02899	1	.002884	1.02706
2	.017161	1.13166	2	.003420	1.13406
3	.004738	3.23972	3	.000948	3.23976
4	.007749	5.39181	4	.001550	5.39197
5	.010789	7.53619	5	.002158	7.53662

Table 2. Eight cases considered here

Space	State Space				Physical Space			
Scheme	I		II		I		II	
Damping	0.2	0.04	0.2	0.04	0.2	0.04	0.2	0.04
Case	A	B	E	F	C	D	G	H

4.1 Effects of Condensation Space

(1) For the damping ratios, the accuracy of Cases A and B is a little lower than Cases C and D for the initial and first approximation. However, the former is much higher than the latter for the rest iterations. This means that the iterative schemes defined in static space are much better than those defined in physical space. Actually, the iterations in physical space can not improve the accuracy of the damping ratios. The damping ratios resulting from the iterative schemes in physical space are usually meaningless because their errors are very larger. The damping ratios of the reduced model with ten iterations, for example, are almost much less than the exact values. Similar conclusion can be drawn from Cases E, F, G, and H. (2) For the first and second damping frequencies, the iterative schemes in state space are better than those in physical space. However, it is not true for the higher three frequencies.

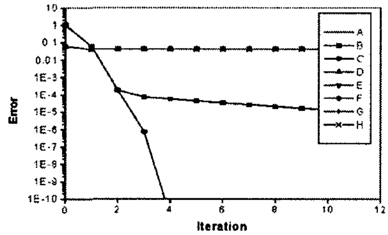
4.2 Effects of Iterative Method

It can be seen from these figures that the Cases E and F are much better than Cases A and B. This means that the errors of the damped frequencies and damping ratios obtained from the subspace iteration (scheme II) defined in state space reduce much more quickly than those from common inverse iteration (scheme I), especially when these results close to the exact.

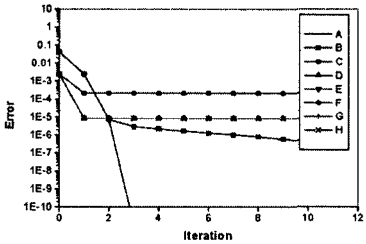
4.3 Effects of Damping

(1) For all the ten figures, Case A is very similar to Case B, and Case E is very similar to Case F. This means that the damping has little effect on the convergence of the two iterative schemes defined in state space. It is true for both the damping ratios and damped frequencies. (2) For the iterative schemes

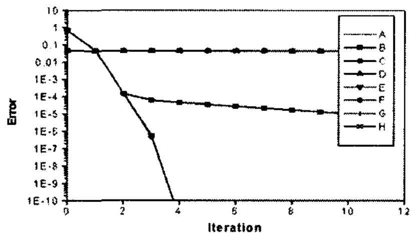
defined in physical space, the accuracy of the damped frequencies for Cases D and H is almost higher than that for Cases C and G.. This shows that the damping has much effect on the convergence of the damped frequencies and that their accuracy increases with the decrease of the damping. However, damping has very little effect on the accuracy of the damping ratio.



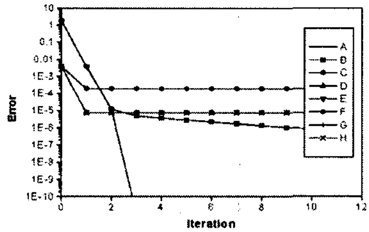
2(a) Errors of the first damping ratio



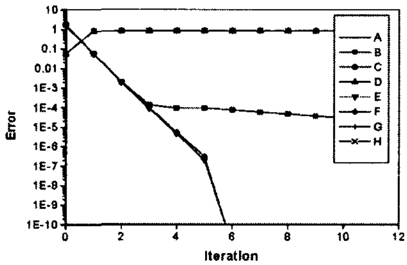
2(b) Errors of the first damping frequency



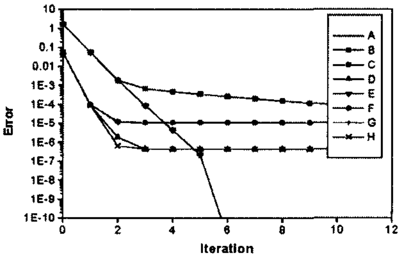
2(c) Errors of the second damping ratio



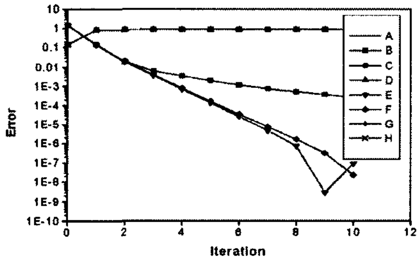
2(d) Errors of the second damping frequency



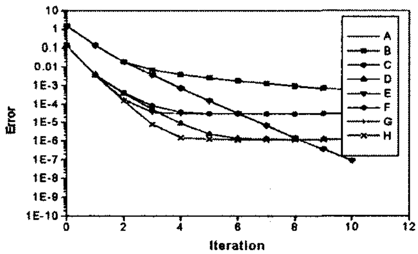
2(e) Errors of the third damping ratio



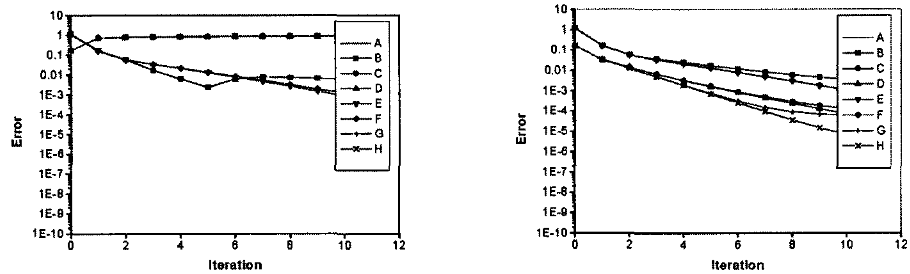
2(f) Errors of the third damping frequency



2(g) Errors of the fourth damping ratio



2(h) Errors of the fourth damping frequency



2(i) Errors of the fifth damping ratio 2(j) Errors of the fifth damping frequency

Figure 1. Errors of damping ratios and frequencies for eight cases

5 Conclusions

Four iterative schemes for dynamic condensation of nonclassically damped models are discussed in this paper. Two of them are defined in state space and the other are defined in physical space. A high building with a tuned mass damper is applied to show the efficiency of these schemes. The effects of dynamic condensation space, condensation schemes, and the damping of the model on the accuracy of the damping ratios and frequencies are discussed by numerical example.

The following three conclusions can be drawn from the discussion.

(1) The iterative schemes defined in state space are much better than those defined in physical space.

(2) The iterative scheme II (subspace iteration) is much better than the scheme I, especially when the reduced model closes to full model.

(3) The damping has little effect on the convergence of the two iterative schemes defined in state space.

Reference

Guyan, R.J. (1965) "Reduction of stiffness and mass matrices", AIAA J, Vol. 3, No. 2, pp. 380.

Irons, B. (1965) "Structural eigenvalue problems-elimination of unwanted variables", AIAA Journal, Vol. 3, pp. 961-962.

Leung, A.T. (1995) "Subspace Iteration for Complex Symmetric Eigenproblem", Journal of Sound and Vibration, Vol. 184, No.4, 1995, pp. 627-637.

Miller, C.A. (1980) "Dynamic Reduction of Structural Models", Journal of Structural Division, Vol. 106, No. 10, pp. 2097-2108.

Mills, J.K. and Ing, J.G.L. (1996) "Dynamic modeling and control of a multi-robot system for assembly of flexible payloads with applications to automotive body assembly", Journal of Robotic Systems, Vol.13, 1996, pp. 817-836.

O'Callahan, J.C. (1989) "A Procedure for an Improved Reduced System (IRS) Model", Proceedings of the 7th International Modal Analysis Conference, Las Vegas, Neveda, USA, pp. 17-21.

Qu, Z.Q., and Fu, A.F. (1998) "New structural dynamic condensation method for finite element models", AIAA Journal, Vol. 36, pp. 1320-1324

Rivera, M.A., Singh, M.P. (1999) and Suarez, L.E., "Dynamic Condensation Approach for Nonclassically Damped Structures," AIAA J, Vol. 37, No. 5, pp. 564-571

Suarez, L.E., and Singh, M.P. (1992) "Dynamic condensation method for structural eigenvalue analysis", AIAA Journal, Vol. 30, pp. 1046-1054.