# A Common Fixed Point Theorem in $M$-Fuzzy Metric Spaces

# $M$- 퍼지거리공간에서의 공통 부동점정리

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**Abstract**

In this paper, using the notion of generalized metric (or D-metric) due to Dhage [3], we give new definition of $M$-fuzzy metric spaces and prove a common fixed point theorem for two mappings under the condition of weak compatible and $R$-weakly commuting mappings in complete $M$-fuzzy metric spaces.

## 1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [20]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology. The concept of fuzzy topology may have very important applications in quantum particle physics particularly in connections with both string and $\varepsilon^{\infty}$ theory which were given and studied by Elnaschie [7,8].

One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of views. In particular, George and Veeramani [11] have introduced and studied a notion of fuzzy metric space with the help of continuous $t$-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalek [14].

In this paper, using the notion of generalized metric (or D-metric) due to Dhage [3], we give new definition of $M$-fuzzy metric spaces and prove a common fixed point theorem for two mappings under the condition of weak compatible and $R$-weakly commuting mappings in complete $M$-fuzzy metric spaces.

**Definition 1.** [3] Let $X$ be a nonempty set. A generalized metric (or D-metric) on $X$ is a function $D: X \to R^{+}$ satisfing the following conditions: for all $x, y, z, a \in X$,

(1) $D(x, y, z) \geq 0$

(2) $D(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D(x, y, z) = D(p\{x, y, z\})$, where $p$ is a permutation function,

(4) $D(x, y, z) \leq D(x, y, a) + D(a, z, z)$.

Then $(X, D)$ is called a generalized metric (or D-metric) space.

Immediate examples of such a function are

$$D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$
$$D(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$

where $d$ is the ordinary metric on $X$.

**Definition 2.** [18] A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a continuous $t$-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a*1 = a$ for all $a\in[0,1]$,

(4) $a*b \le c*d$ whenever $a\le c$ and $b \le d$, for each $a, b, c, d\in[0,1]$.

Two typical examples of continuous $t$-norm are $a*b = ab$ and $a*b=\min\{a, b\}$.

**Definition 3.** The 3-tuple $(X, M, *)$ is called a $M$-fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t$-norm and $M$ are fuzzy sets on $X^3\times(0,\infty)$ satisfying the following conditions: for all $x, y, z, a\in X$ and $t, s > 0$,

(1) $M(x, y, z, t) > 0$;

(2) $M(x, y, z, t) = 1$ if and only if $x = y = z$;

(3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, where $p$ is a permutation function;

(4) $M(x, y, a, t) * M(a, z, z, s) \le M(x, y, z, t+s)$;

(5) $M(x, y, z, \cdot): (0,\infty)\to[0,1]$ is continuous.

**Lemma 1.** Let $(X, M, *)$ be a $M$-fuzzy metric space. For any $x, y, z\in X$ and $t>0$, we have

(1) $M(x, x, y, t) = M(x, y, y, t)$.

(2) $M(x, y, z, \cdot)$ is nondecreasing.

In the following examples, we know that both D-metric and fuzzy metric induce a $M$-fuzzy metric.

**Example 1.** Let $(X, D)$ be a D-metric space. Denote $a*b = ab$ for all $a, b\in[0,1]$. For each $t>0$, let

$$M(x, y, z, t) = \frac{t}{t + D(x, y, z)},$$

for all $x, y, z\in X$. Then $(X, M, *)$ is a $M$-fuzzy metric space. We call the $M$-fuzzy metric $M$ induced by the metric D the standard $M$-fuzzy metric.

**Example 2.** Let $(X, N, *)$ be a fuzzy metric space. Define $M: X^3\times(0,\infty)\to[0,1]$ by

$$M(x, y, z, t) = N(x, y, t) * N(y, z, t) * N(z, x, t)$$

for every $x, y, z\in X$. Then $(X, M, *)$ is a $M$-fuzzy metric space.

**Definition 4.** Let $(X, M, *)$ be a $M$-fuzzy metric space and $\{x_n\}$ be a sequence in $X$.

(1) $\{x_n\}$ is said to be convergent to $x$ ($\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} M(x, x, x_n, t) = 1$ for all $t>0$.

(2) $\{x_n\}$ is called a Cauchy sequence if for each $0<\varepsilon<1$ and $t>0$, there exists $n_0\in N$ such that $M(x_n, x_n, x_m, t) > 1-\varepsilon$ for all $n, m\ge n_0$.

(3) A $M$-fuzzy metric in which every Cauchy sequence is convergent is said to be complete.

## 2. Main Results

**Lemma 2.** Let $(X, M, *)$ be a $M$-fuzzy metric space. Then $M$ is continuous function on $X^3\times(0,\infty)$.

**Lemma 3.** Let $(X, M, *)$ be a $M$-fuzzy metric space. If we define $E_\lambda: X^3\to R^+\cup\{0\}$ by

$$E_\lambda(x, y, z) = \inf\{t>0: M(x, y, z, t) > 1-\lambda\}$$

for every $\lambda\in(0,1)$, then

(1) for each $\mu\in(0,1)$, there exists $\lambda\in(0,1)$ such that

$$E_\mu(x_1, x_1, x_n) \le E_\lambda(x_1, x_1, x_2) + E_\lambda(x_2, x_2, x_3) + \cdots + E_\lambda(x_{n-1}, x_{n-1}, x_n)$$

for any $x_1, x_2, \cdots, x_n\in X$.

(2) a sequence $\{x_n\}$ is convergent to $x$ in $(X, M, *)$ if and only if $E_\lambda(x_n, x_n, x)\to 0$. Also, $\{x_n\}$ is Cauchy sequence if and only if it is Cauchy with respect to $E_\lambda$.

**Lemma 4.** Let $(X, M, *)$ be a $M$-fuzzy metric space. If there is a number $k>1$ such that

$$M(x_n, x_n, x_{n+1}, t) \geq M(x_0, x_0, x_1, k^n t)$$

for all $t>0$ and $n \in N$, then sequence $\{x_n\}$ is a Cauchy sequence.

**Definition 5.** Let $A$ and $S$ be mappings from a $M$-fuzzy metric space $(X, M, *)$ into itself. Then the mappings $A$ and $S$ are said to be weak compatible if they commute at a coincidence point, that is, $Ax = Sx$ implies $ASx = ASx$.

Let $\Psi$ denotes a family of mappings such that each $\phi \in \Psi$, $\phi:[0,1] \to [0,1]$ is a continuous function such that $\phi(s) > s$ for every $s \in [0,1)$.

**Theorem 1.** Let $S, T$ be self-mappings of a complete $M$-fuzzy metric space $(X, M, *)$ satisfying the following conditions:

(1) $S(X) \subset T(X)$ and $S(X)$ or $T(X)$ is a closed subset of $X$,

(2) $M(Sx, Sy, Sz, t) \geq \phi(M(Tx, Ty, Tz, kt))$ for every $x, y, z \in X$, $k>1$ and $\phi \in \Psi$,

(3) the pair $(S, T)$ are weak compatible.

Then $S, T$ have a unique common fixed point in $X$.

**Theorem 2.** Let $S, T$ be self-mappings of a complete $M$-fuzzy metric space $(X, M, *)$ satisfying the following conditions:

(1) $S(X) \subset T(X)$ and $S(X)$ or $T(X)$ is a closed subset of $X$,

(2)

$$M(Sx, Sy, Sz, t) \geq \phi\left(\min \begin{Bmatrix} M(Tx, Ty, Tz, kt), & M(Sx, Tx, Tx, kt) \\ M(Sx, Ty, Ty, kt), & M(Sx, Tz, Tz, kt) \\ M(Sy, Ty, Tz, kt), & M(Sy, Ty, Ty, kt) \\ M(Sy, Tz, Tz, kt), & M(Sz, Ty, Tz, kt) \\ M(Sz, Ty, Ty, kt), & M(Sz, Tz, Tz, kt) \end{Bmatrix}\right)$$

for every $x, y, z \in X$, $k>1$ and $\phi \in \Psi$,

(3) the pair $(S, T)$ are weak compatible.

Then $S, T$ have a unique common fixed point in $X$.

**Definition 6.** Let $f$ and $g$ be maps from a $M$-fuzzy metric space $(X, M, *)$ into itself. The maps $f$ and $g$ are said to be $R$-weakly commuting if there exists some positive real number $R$ such that

$$M(fgx, gfx, gfx, t) \geq M(fx, gx, gx, \frac{t}{R})$$

for each $x \in X$ and $t > 0$.

**Theorem 3.** Let $(X, M, *)$ be a complete $M$-fuzzy metric space and let $f$ and $g$ be $R$-weakly commuting self-mappings of $X$ satisfying the following conditions:

(1) $f(X) \subset g(X)$,

(2) $f$ or $g$ is continuous,

(3) $M(fx, fy, fz, t) \geq \gamma(M(gx, gy, gz, t))$,

where $\gamma:[0,1] \to [0,1]$ is a continuous mapping such that $\gamma(a) > a$ for each $a \in (0,1)$.

Then $f$ and $g$ have a unique common fixed point in $X$.

Now we give an example to support our Theorem 3.

**Example 3.** Consider Example 1 in which $X = [0,1]$. Define $f(x) = 1$ and

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

on $X$. It is evident that $f(X) \subset g(X)$, $f$ is continuous and $g$ is discontinuous. Define $\gamma:[0,1] \to [0,1]$ by $\gamma(a) = \sqrt{a}$, then

$$M(fx, fy, fy, t) \geq \gamma(M(gx, gy, gy, t))$$

for all $x, y \in X$ and $f$ and $g$ are $R$-weakly commuting. Thus all the conditions of last theorem are satisfied and 1 is a common fixed point of $f$ and $g$.

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