최소-Dilation 신장 트리 찾기의 NP-hard 증명

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Computing a Minimum-Dilation Spanning Tree is NP-hard

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요 약

Given a set S of n points in the plane, a minimum-dilation spanning tree of S is a tree with vertex set S of smallest possible dilation. We show that given a set S of n points and a dilation $\delta > 1$, it is NP-hard to determine whether a spanning tree of S with dilation at most δ exists.

1. Introduction

A *geometric network* is a weighted undirected graph whose vertices are points in $\mathbb{R}^d$, and in which the weight of an edge is the Euclidean distance between its endpoints. Geometric networks have many applications: most naturally, many communication networks (road networks, railway networks, telephone networks) can be modelled as geometric networks.

In a geometric network $G = (S, E)$ on a set S of n points, the *graph distance* $d_G(u, v)$ of $u, v \in G$ is the length of a shortest path from u to v in G. Some applications require a geometric network for a given set S of points that includes a relatively short path between every two points in S. More precisely, we consider the factor by which the graph distance $d_G(u, v)$ differs from the Euclidian distance $|uv|$. This factor is called the *dilation* Δ of the pair (u, v) in G, and is formally expressed as:

$$\Delta_G(u, v) := \frac{d_G(u, v)}{|uv|}$$

The dilation or stretch factor $\Delta(G)$ of a graph is the maximum dilation over all vertex pairs:

$$\Delta(G) := \max_{\substack{u,v\in S\\ u\neq v}} \Delta_G(u, v) = \max_{\substack{u,v\in S\\ u\neq v}} \frac{d_G(u, v)}{|uv|}.$$

A network G is called a t-spanner if $\Delta(G) \leqslant t$.

The complete graph has optimal dilation and is easy to compute, but for many applications its high cost is unacceptable. Therefore one usually seeks to construct networks that do not only have small dilation, but also have a low number of edges. Therefore there has also been considerable interest from a theoretical perspective [4, 10]. Several algorithms have been published to compute a $(1 + \varepsilon)$-spanner with $O(n)$ edges for any given set of points S [2, 8, 9, 11] and any $\varepsilon > 0$. Farshi and Gudmundsson did an experimental study of such algorithms [5].

Although the number of edges in the spanners from these algorithms is linear in n, it can still be rather large due to the hidden constants in the O-notation that depend on ε and the dimension d. Therefore there has also been attention to the problem with the priorities reversed: given a certain number of edges, how small a dilation can we realize? Das and Heffernan [3] showed how to compute in $O(n \log n)$ time, for any constant $\varepsilon' > 0$, a spanner with $(1 + \varepsilon')n$ edges, degree three, and constant dilation in the sense that it only depends on ε' and d. Aronov et al. [1] studied even smaller spanners and showed how to compute in $O(n \log n)$ time, for any constant $k > 0$, a spanner with $n-1+k$ edges and dilation $O(n/(k+1))$. They proved that this is optimal in the worst case. With this work Aronov et al. provided a generalization (and better constants) of Eppstein's observation that any spanning tree on the vertices of a regular polygon has dilation $\Omega(n)$, and that the minimum-weight spanning tree achieves dilation $O(n)$ for any point set S [4]. Note that any spanner must at least connect the n points of S, and therefore include a spanning tree of S—spanners with fewer than $n - 1$ edges are impossible. Eppstein posed the following questions:

> Is it possible to construct the exact minimum-dilation geometric spanning tree, or an approximation to it, in polynomial time? Does the minimum-dilation spanning tree have any edge crossings?

Very recently these related questions also got the attention of Gudmundsson and Smid and of Klein and Kutz. Gudmundsson and Smid [6] found by reduction from 3SAT that, given a geometric graph G', a dilation δ and a number $k \geqslant n - 1$, it is NP-hard to decide whether G contains a δ-spanner with at most k edges. Klein and Kutz [7] are about to publish a proof that, given a set of points P in the plane, a dilation δ and a number $k \geqslant n - 1$, it is NP-hard to decide whether there is a plane δ-spanner with at most k edges. In both cases the proofs are based on instances of the problem with $k > n-1$, and the question whether a spanning *tree* with dilation at most δ can be found in polynomial time remains open.

Klein and Kutz answer Eppstein's second question by providing a set of seven points whose minimum-dilation spanning tree has edge crossings.

Our results We first address Eppstein's second question: we show that even the minimum-dilation spanning tree of a set of only five points may have edge crossings. Sets of less than five points always have a minimum-dilation spanning tree without edge crossings.

We then give a negative answer to the first question: given a set S of n points in the plane and a dilation $\delta > 1$, it is NP-hard to decide whether a spanning tree of S with dilation at most δ exists—regardless if edge crossings are allowed or not. Thus the problems studied by Gudmundsson and Smid and by Klein and Kutz remain NP-hard even if the number of edges k is restricted to $n-1$.

Our NP-hardness proof[1] is a reduction from SETPARTITION:

> SETPARTITION
> Given a set $V = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ of n positive integers, is there a partition of V into subsets A and A' such that $A \cap A' = \emptyset$, $A \cup A' = V$ and $\sum_{\alpha \in A} \alpha = \sum_{\alpha \in A'} \alpha$?

We show that a set V of n positive integers can be transformed, in polynomial time, to $8n+8$ points in the plane, such that there exists a partition of V if and only if there exists a geometric spanning tree $\mathcal{T}$ on S with $\Delta(\mathcal{T}) \leqslant 3/2$. Conceptually, our construction is quite simple, the difficulty being to ensure that no unwanted solutions or interactions can arise.

To prove NP-hardness of the problem, we have to do the reduction on a Turing-machine or an equivalent model, and so our spanning tree problem should take integer coordinates as input. Our construction does not quite achieve this: we construct points as the intersection of circles, and therefore with irrational coordinates. We solve this problem by showing that if the coordinates of these points are *approximated* by rational points with precision polynomial in the input size, the construction still goes through, achieving a polynomial-time reduction to a problem of polynomial size. For reasons of space, we omit the discussion of the approximation step from this extended abstract.

2. Minimum-dilation spanning trees with edge crossings

Suppose we are given a set S of points. Klein and Kutz [7] have an example where $|S| = 7$ and the minimum-dilation spanning tree of S has edge crossings. Aronov et al. [1] already observed that minimum-dilation spanning *paths* may have edge crossings. Below we give an example with $|S| = 5$, and prove that there is no smaller set S that does not have a crossing-free minimum-dilation spanning tree.

For $u, v \in S$, we call uv a δ-critical edge if for every point $w \in S \setminus \{u, v\}$ we have

$$\delta \cdot |uv| < |uw| + |wv|.$$

Clearly, any spanning tree $\mathcal{T}$ of S that does not include all δ-critical edges has dilation $\Delta(\mathcal{T}) > \delta$.

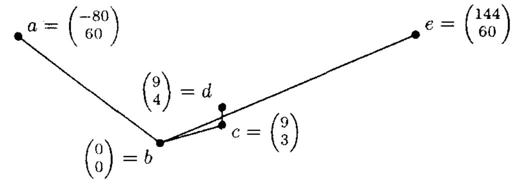

Figure 1: A set of five points whose minimum-dilation spanning tree has dilation 8/7 and has an edge crossing (not to scale).

Figure 1 shows a set of five points $S = \{a, b, c, d, e\}$. The reader may verify that the minimum-dilation spanning tree of S consists of the edges ab, bc, cd and be, where cd and be intersect.

Theorem 1 *For $n \geqslant 5$, there are sets of n points in the plane that do not have a minimum-dilation spanning tree without edge crossings. For $n < 5$, every set of n points in $\mathbb{R}^d$ has a minimum-dilation spanning tree without edge crossings.*

Proof. For $n = 5$, an example is given in Figure 1. The example can easily be extended with additional points.

For $n < 5$, observe that intersections between possible edges are possible only if $n = 4$ and the points are co-planar and in convex position. Suppose $\mathcal{T}$ is a minimum-dilation spanning tree with an edge crossing on four such points a, b, c, d. Without loss of generality, assume ad and bc are the intersecting edges, cd is the third edge, and b lies closer to d than to c. We now create another spanning tree $\mathcal{T}'$ by taking $\mathcal{T}$ and replacing edge bc by edge bd. This increases only $d_{\mathcal{T}}(b, c)$. Hence we get:

$$\Delta(\mathcal{T}') = \max\left\{\Delta(\mathcal{T}), \frac{d_{\mathcal{T}'}(b,c)}{|bc|}\right\}$$
$$< \max\left\{\Delta(\mathcal{T}), \frac{d_{\mathcal{T}}(b,d)}{|bd|}\right\} = \Delta(\mathcal{T}).$$

So $\mathcal{T}'$ is a minimum-dilation spanning tree of a, b, c and d without edge crossings. ∎

3. Computing a minimum-dilation tree is NP-hard

We give a reduction from SETPARTITION. In Section 3.1, we first show how to transform a set V of n positive integers into a set S of $8n+8$ points, and prove some properties of this set. We say that $\Delta(S) = \delta$ if and only if S admits a spanning tree $\mathcal{T}$ with $\Delta(\mathcal{T}) \leqslant \delta$. In Section 3.2 we show that if $\Delta(S) \leqslant 3/2$, then the set partition problem has a positive answer. In Section 3.3, we show that if the set partition problem has a positive answer, then $\Delta(S) \leqslant 3/2$, and there is a spanning tree with dilation 3/2 on S that does not have any edge crossings. In Section 3.4, we explain how to replace the irrational coordinates in our construction by rational coordinates of polynomial size, so that they can be handled on a Turing machine. Thus we prove the following:

[1] Note that we cannot claim NP-completeness of the problem, as it is not known how to do the necessary distance computations involving sums of square roots in NP.

Theorem 2 *Given a set S of n points in the plane and a dilation $\delta > 1$, it is NP-hard to determine whether a geometric spanning tree of S with dilation at most δ exists. It is also NP-hard to determine whether there is a geometric spanning tree of S with dilation at most δ and without edge crossings.*

3.1 Construction of S

We are given a set of n positive integers, which we scale to have sum $1/10$. Hence our input consists of a set of positive rational numbers $V = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ with $\sigma = \sum_{1 \leqslant i \leqslant n} \alpha_i = 1/10$. Figure 2 shows the general structure of the construction. It is symmetric around the y-axis, and so we only need to describe the right half of the construction.

We create $3n + 1$ points lying on the line with slope $3/4$ through the point $(5/2, 0)$:

$$a_i = \begin{pmatrix} 5/2 \\ 0 \end{pmatrix} + (4^{i-1} - 1)\begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \forall 1 \leqslant i \leqslant n+1$$

$$b_i = a_i + \frac{4^{i-1}}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \forall 1 \leqslant i \leqslant n$$

$$c_i = b_i + \frac{3 \cdot 4^{i-1}}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \forall 1 \leqslant i \leqslant n$$

The distances between these points are as follows:

$$\begin{aligned} |a_i a_{i+1}| &= 15 \cdot 4^{i-1} \\ |a_i b_i| &= 1 \cdot 4^{i-1} \\ |b_i c_i| &= 3 \cdot 4^{i-1} \\ |c_i a_{i+1}| &= 11 \cdot 4^{i-1} \\ |a_1 a_{n+1}| &= 5(4^n - 1) \end{aligned}$$

All these points have rational coordinates, and can be expressed with a number of bits polynomial in the number n of input points. We now add another n points d_i, for $1 \leqslant i \leqslant n$, making use of the input numbers α_i. These points lie slightly above the line $a_1 a_{n+1}$, and are defined by the two equations:

$$\begin{aligned} |d_i a_{i+1}| &= 2 \cdot 4^{i-1} \\ |c_i d_i| &= 9 \cdot 4^{i-1} + \alpha_i \end{aligned}$$

Figure 3 shows the interval between a_i and a_{i+1}. Since

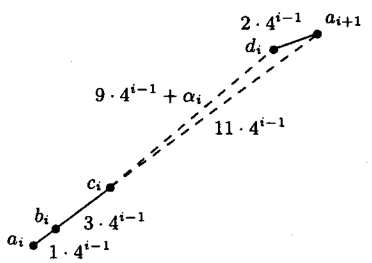

Figure 3: The construction between a_i and a_{i+1}

$|c_i a_{i+1}| = 11 \cdot 4^{i-1}$ and $0 < \alpha_i \leqslant 1/10$, it is clear that d_i exists. We add two more points at the far end:

$$p_1 = a_{n+1} + \left(\frac{4^n}{9} - \frac{179}{1800}\right)\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$p_2 = a_{n+1} + 4\left(\frac{4^n}{9} - \frac{179}{1800}\right)\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Both points lie on the line through a_{n+1} with slope $-4/3$, and so $\angle a_1 a_{n+1} p_2$ is a right angle. We have

$$|a_{n+1} p_1| = \frac{5}{9} 4^n - \frac{179}{360}$$

$$|a_{n+1} p_2| = 4|a_{n+1} p_1| = \frac{5}{9} 4^{n+1} - \frac{179}{90}$$

We denote the mirror images under reflection in the y-axis of the $4n + 3$ points a_i, b_i, c_i, d_i, p_i constructed so far as $a'_i, b'_i, c'_i, d'_i, p'_i$. Our point set S consists of $8n + 8$ points, namely the original points, their mirror images, and two more points on the y-axis:

$$q_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$q_2 = \begin{pmatrix} 0 \\ -\frac{25}{9} 4^n + \frac{11}{18} \end{pmatrix}$$

We have

$$p_2 - q_2 = \left(\frac{4^{n+1}}{3} - \frac{101}{150}\right)\begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

so $q_2 p_2$ is parallel to $a_1 a_{n+1}$, and

$$|q_2 p_2| = |q_2 p'_2| = \frac{5}{3} \cdot 4^{n+1} - \frac{101}{30}.$$

We now prove some basic properties of the constructed point set S.

Lemma 1 *We have $\cos \angle c_i a_{i+1} d_i > 1 - 4^{1-i}/22 \geqslant 21/22$, and the y-coordinate of d_i is strictly smaller than the y-coordinate of a_{i+1}, for $1 \leqslant i \leqslant n$.*

Proof. Since $\alpha_i \leqslant \frac{1}{10}$, the reader may prove by using the cosine theorem. □

For $u, v \in S$, we call uv a *critical edge* if for every $w \in S \setminus \{u, v\}$ we have

$$(3/2)|uv| < |uw| + |wv|.$$

As we observed in the previous section, any spanning tree $\mathcal{T}$ on S that does not include a critical edge uv must have dilation $\Delta(\mathcal{T}) > 3/2$. Let us call the point $w \in S \setminus \{u, v\}$ minimizing the sum $|uw| + |wv|$ the *nearest neighbor* of uv.

Lemma 2 *The following edges are all critical: $q_1 a_1$, $a_{n+1} p_1$, $p_1 p_2$, $a_i b_i$, $b_i c_i$, $d_i a_{i+1}$ (where $1 \leqslant i \leqslant n$), and their mirror images.*

Proof. We omit the proof in this short version. □

The enumeration in Lemma 2 is exhaustive: these are all the critical edges. However, to form the connection between c_i and d_i, only two choices are possible—this is the choice at the heart of our NP-hardness argument.

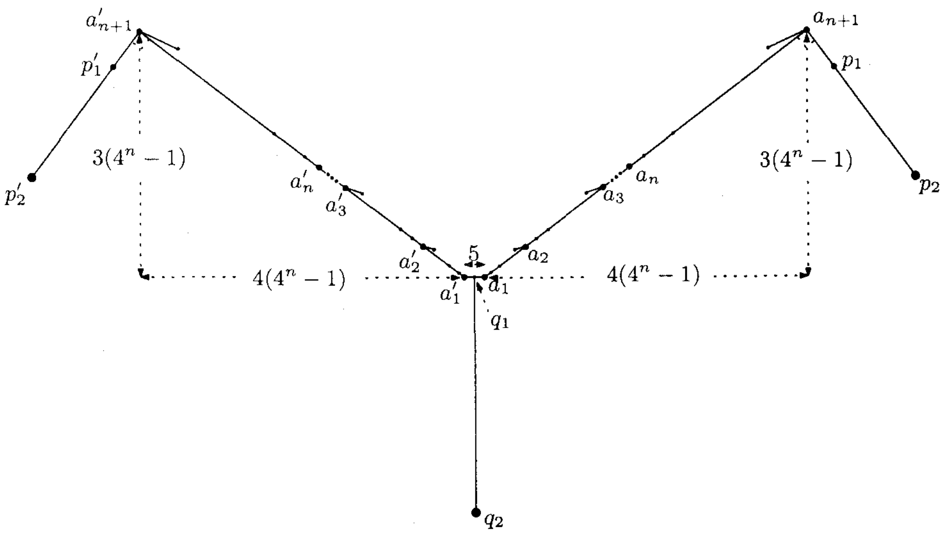

Figure 2: Construction of S

Lemma 3 *If $\mathcal{T}$ is a spanning tree on S with $\Delta(\mathcal{T}) \leqslant 3/2$, then it contains exactly one of the edges $c_i d_i$ and $c_i a_{i+1}$, and exactly one of the edges $c'_i d'_i$ and $c'_i a'_{i+1}$, for each $1 \leqslant i \leqslant n$.*

Proof. Consider points $c_i d_i$, for some $1 \leqslant i \leqslant n$. If $\mathcal{T}$ contains neither $c_i d_i$ nor $c_i a_{i+1}$, then the shortest path from c_i to d_i in $\mathcal{T}$ must make use of a point $w \in S \setminus \{c_i, d_i, a_{i+1}\}$, and its length is at least $|c_i w| + |w d_i|$. The point w minimizing this expression is b_i, but since

$$|d_i b_i| + |b_i c_i| > (11 + 3 - 2)4^{i-1} + 3 \cdot 4^{i-1} > \tfrac{3}{2}|c_i d_i|,$$

this is not good enough. It follows that $\mathcal{T}$ must contain at least one of the edges $c_i d_i$ or $c_i a_{i+1}$. Since by Lemma 2 it also contains $d_i a_{i+1}$, it cannot contain both edges. □

3.2 If $\Delta(S) \leqslant 3/2$ then there is a set partition

We assume in this section that a spanning tree $\mathcal{T}$ on S exists with $\Delta(\mathcal{T}) \leqslant 3/2$. Our aim is to show that the SETPARTITION problem on V has a positive solution.

We define the set partition as follows:

$$A := \{\alpha_i \in V \mid \mathcal{T} \text{ contains } c_i d_i\},$$
$$A' := \{\alpha_i \in V \mid \mathcal{T} \text{ contains } c'_i d'_i\}.$$

Let $\sigma_A = \sum_{\alpha \in A} \alpha$ and $\sigma_{A'} = \sum_{\alpha \in A'} \alpha$. We need to show that $\sigma_A = \sigma_{A'}$, that $A \cap A' = \emptyset$, and that $A \cup A' = V$.

Lemma 4 *We have $A \cup A' = V$.*

Proof. Let us assume that for some $1 \leqslant i \leqslant n$, neither $c_i d_i$ nor $c'_i d'_i$ is in $\mathcal{T}$. We consider the dilation of the pair $d'_i d_i$. The shortest path from d_i to d'_i in $\mathcal{T}$ must go through both a_{i+1} and a'_{i+1}, and so its length is at least

$$\begin{aligned} d_{\mathcal{T}}(d'_i, d_i) &\geqslant 2|d_i a_{i+1}| + 2|a_1 a_{i+1}| + |a_1 a'_1| \\ &= 4 \cdot 4^{i-1} + 10(4^i - 1) + 5 \\ &= 11 \cdot 4^i - 5. \end{aligned}$$

On the other hand, $|d'_i d_i| = |a'_{i+1} a_{i+1}| - 2\ell$, where ℓ is the length of the projection of $d_i a_{i+1}$ on the x-axis. By Lemma 1, we have $\ell > \frac{4}{5}|d_i a_{i+1}|$, and so

$$\begin{aligned} |d'_i d_i| &= |a'_{i+1} a_{i+1}| - 2\ell \\ &< |a'_{i+1} a_{i+1}| - \tfrac{8}{5}|d_i a_{i+1}| \\ &= 7.2 \cdot 4^i - 3. \end{aligned}$$

$$d_{\mathcal{T}}(d'_i, d_i)/|d'_i d_i| > (11 \cdot 4^i - 5)/(7.2 \cdot 4^i - 3) > 3/2.$$

This is a contradiction, so no such i can exist, and the lemma follows. □

Lemma 5 *We have $A \cap A' = \emptyset$ and $\sigma_A = \sigma_{A'} = 1/20$.*

Proof. The spanning tree $\mathcal{T}$ must contain the $6n + 6$ critical edges enumerated in Lemma 2. By Lemma 3, it must also contain n edges connecting each c_i to either d_i or a_{i+1}, and by symmetry also n edges connecting each c'_i to either d'_i or a'_{i+1}. Since S consists of $8n + 8$ points, $\mathcal{T}$ has $8n + 7$ edges, and so there is only one edge unaccounted for. This edge must connect q_2 to some point $q \in S \setminus \{q_2\}$. We note that $|q_2 q| \geqslant \frac{25}{9}4^n - \frac{11}{18}$, where equality holds only if $q = q_1$ (see Figure 2).

Since $\Delta(\mathcal{T}) \leqslant 3/2$, we have

$$\begin{aligned} & d_{\mathcal{T}}(p'_2, q_2) + d_{\mathcal{T}}(q_2, p_2) \\ &\qquad \leqslant \frac{3}{2}(|p'_2 q_2| + |q_2 p_2|) = 5 \cdot 4^{n+1} - \frac{101}{10}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & d_{\mathcal{T}}(p'_2, q_2) + d_{\mathcal{T}}(q_2, p_2) \\ &= d_{\mathcal{T}}(p'_2, q) + |q q_2| + |q_2 q| + d_{\mathcal{T}}(q, p_2) \\ &\geqslant d_{\mathcal{T}}(p'_2, p_2) + 2|q q_2| \\ &\geqslant d_{\mathcal{T}}(p'_2, p_2) + \frac{50}{9}4^n - \frac{11}{9}, \end{aligned}$$

where equality holds only if $q = q_1$. Now,

$$\begin{aligned} d_{\mathcal{T}}(p_2', p_2) &= |p_2' a_{n+1}'| + d_{\mathcal{T}}(a_{n+1}', a_1') \\ &\quad + |a_1' a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1} p_2| \\ &= \frac{10}{9} 4^{n+1} + \frac{46}{45} + d_{\mathcal{T}}(a_1', a_{n+1}') + d_{\mathcal{T}}(a_1, a_{n+1}) \\ &= \frac{130}{9} 4^n - \frac{404}{45} + \sigma_A + \sigma_{A'}. \end{aligned}$$

Since $A \cup A' = V$ by Lemma 4, we have $\sigma_A + \sigma_{A'} \geqslant 1/10$, where equality holds only if $A \cap A' = \emptyset$. Putting everything together we get

$$\begin{aligned} 5 \cdot 4^{n+1} - \frac{101}{10} &\geqslant d_{\mathcal{T}}(p_2', q_2) + d_{\mathcal{T}}(q_2, p_2) \\ &\geqslant d_{\mathcal{T}}(p_2', p_2) + \frac{50}{9} 4^n - \frac{11}{9} \\ &\geqslant 5 \cdot 4^{n+1} - \frac{102}{10} + \sigma_A + \sigma_{A'} \\ &\geqslant 5 \cdot 4^{n+1} - \frac{101}{10}. \end{aligned}$$

Since equality holds, we must have $A \cap A' = \emptyset$ and $q = q_1$. By using $d_{\mathcal{T}}(q_2, p_2) \leqslant \frac{3}{2}|q_2 p_2| = 10 \cdot 4^n - \frac{101}{20}$ and $d_{\mathcal{T}}(q_2, p_2) = |q_2 q_1| + |q_1 a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1} p_2|$, we get $\sigma_A \leqslant 1/20$. Since similarly we have $\sigma_{A'} \leqslant 1/20$ and $\sigma_A + \sigma_{A'} = 1/10$, it follows that $\sigma_A = \sigma_{A'} = 1/20$. ◻

3.3 If a set partition exists, then $\Delta(S) \leqslant 3/2$

Let us call a tree $\mathcal{T}$ on S a *standard tree* if it consists of the critical edges, the edge $q_1 q_2$, and for each $1 \leqslant i \leqslant n$ either $c_i d_i$ or $c_i a_{i+1}$ and either $c_i' d_i'$ or $c_i' a_{i+1}'$. In the following lemmas we will show that *any* standard tree has dilation less than 3/2 for nearly all pairs of points in S, excluding only the pairs (d_i', d_i) (for $1 \leqslant i \leqslant n$), (q_2, p_2), and (q_2, p_2'). These remaining pairs are where the existence of a set partition is critical.

Let $\mathcal{T}$ be an arbitrary standard tree. Let H be the set of points of S to the right of the y-axis, except p_1 and p_2. Symmetrically, let H' be the set of points of S to the left of the y-axis, except p_1' and p_2'.

$$H := \{a_i, b_j, c_j, d_j \mid 1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n\}$$
$$H' := \{a_i', b_j', c_j', d_j' \mid 1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n\}$$

Below, in Lemma 6 and 7 we first prove that the dilation on paths within $H \cup \{q_1\}$ is less than 3/2. By symmetry, these lemmas also apply to paths within $H' \cup \{q_1\}$. Next, in Lemma 8, 9, and 10, we analyse the dilation on paths between H and H', except paths from d_i' to d_i (for $1 \leqslant i \leqslant n$) such that $\mathcal{T}$ contains neither $c_i d_i$ nor $c_i' d_i$. Lemma 11 deals with paths from $\{p_2', p_1', p_1, p_2\}$ to $\{p_2', p_1'\} \cup H' \cup \{q_1\} \cup H \cup \{p_1, p_2\} = S \setminus \{q_2\}$. It remains to consider the dilation on pairs that involve $\{q_2\}$: Lemma 12 treats this case, except for the pairs (q_2, p_2) and (q_2, p_2'). We then show in Lemma 13 that if an equal partition of V exists, we can get dilation at most 3/2 also on (q_2, p_2), (q_2, p_2') and on all pairs (d_i', d_i) (for $1 \leqslant i \leqslant n$). Thus we prove that if a set partition exists, $\Delta(S) \leqslant 3/2$.

To facilitate the analysis, we use the following notation: denote by $w^{\downarrow}$ the orthogonal projection of w on the line through a_1 and a_{n+1}. Let $P_{\mathcal{T}}(u, v)$ be the path from u to v in $\mathcal{T}$. The edges and vertices of the path may depend on the choice of $\mathcal{T}$: for example, d_i lies on $P_{\mathcal{T}}(a_1, a_{n+1})$ if and only if $\mathcal{T}$ contains $c_i d_i$.

We first concentrate on the dilation between points a_i, b_i, c_i and d_i in one half of the tree.

Lemma 6 *For any pair of points (not necessarily vertices) $u, v \in P_{\mathcal{T}}(a_{n+1}, w)$, where $w \in H$, we have $d_{\mathcal{T}}(u, v) < (22/21)|u^{\downarrow} v^{\downarrow}|$ and $\Delta_{\mathcal{T}}(u, v) < 22/21 < 3/2$.*

Lemma 7 *For any pair of vertices $u, v \in H \cup \{q_1\}$, we have $\Delta_{\mathcal{T}}(u, v) < 3/2$.*

We omit the proofs of Lemma 6 and 7 in this short version.

In the following three lemmas we turn our attention to pairs of points in opposite halves of the tree (still excluding p_1, p_2, p_1', p_2' and q_2). We also omit the proofs for lack of space.

Lemma 8 *For any pair of points (not necessarily vertices) $u, v \in P_{\mathcal{T}}(a_{n+1}, a_{n+1}')$, we have $\Delta_{\mathcal{T}}(u, v) < 91/68 < 3/2$.*

Lemma 9 *For any pair of points (d_i', u), where $1 \leqslant i \leqslant n$ and u is a point (not necessarily a vertex) on $P_{\mathcal{T}}(a_{n+1}, a_1)$, we have $\Delta_{\mathcal{T}}(d_i', u) < 3/2$.*

Note that the above Lemma applies symmetrically to pairs of points (d_i, u) where $1 \leqslant i \leqslant n$ and u is a point (not necessarily a vertex) on $P_{\mathcal{T}}(a_{n+1}', a_1')$.

Lemma 10 *For any pair of vertices d_i', d_j with $1 \leqslant i, j \leqslant n$ and $i \neq j$, we have $\Delta_{\mathcal{T}}(d_i', d_j) < 3/2$.*

We now study pairs of vertices involving p_1, p_2, p_1' and/or p_2', but not q_2.

Lemma 11 *For any pair of vertices u, v where $u \in \{p_1, p_2, p_1', p_2'\}$ and $v \in S \setminus \{q_2\}$, we have $\Delta_{\mathcal{T}}(u, v) < 3/2$.*

Proof. We assume that $u \in \{p_1, p_2\}$ (the case of $u \in \{p_1', p_2'\}$ is symmetric). We now distinguish four cases for v: first $v \in \{p_1, p_2, a_{n+1}\}$, second $v \in H \setminus \{a_{n+1}\}$, third $v \in H' \cup \{q_1\}$, and fourth $v \in \{p_1', p_2'\}$.

First, if $v \in \{p_1, p_2, a_{n+1}\}$, then the connection between u and v is a straight line and the dilation is 1.

Second, if $v \in H \setminus \{a_{n+1}\}$, then the path from u to v goes through a_{n+1}, and $\angle u a_{n+1} v \geqslant \pi/2$. Hence the dilation for the pair (u, v) is (using Lemma 6):

$$\frac{d_{\mathcal{T}}(u, v)}{|uv|} < \frac{|u a_{n+1}| + \frac{22}{21}|a_{n+1} v|}{(|u a_{n+1}| + |a_{n+1} v|)/\sqrt{2}} < \frac{22}{21}\sqrt{2} < \frac{3}{2}.$$

Third, if $v \in H' \cup \{q_1\}$, let w be a point where the segment uv intersects the path from a_{n+1} to a_1 (which is a part of the path from u to v). By the analysis of the previous case $\Delta_{\mathcal{T}}(u, w) < 3/2$, and by Lemma 8 or 9 we have $\Delta_{\mathcal{T}}(w, v) < 3/2$; hence $\Delta_{\mathcal{T}}(u, v) < 3/2$.

Finally, if $v \in \{p_1', p_2'\}$, let w be defined as above, and let w' be a point where the segment uv intersects the path from a_1' to a_{n+1}'. By the analysis of the second case $\Delta_{\mathcal{T}}(u, w) < 3/2$ and $\Delta_{\mathcal{T}}(w', v) < 3/2$, and by Lemma 8 we have $\Delta_{\mathcal{T}}(w, w') < 3/2$; hence $\Delta_{\mathcal{T}}(u, v) < 3/2$. ◻

It remains to consider the dilation on pairs of points that involve q_2. We only consider the dilation on pairs of points (u, q_2) where $u \notin \{p_2, p_2'\}$: the dilation of (p_2, q_2) and (p_2', q_2) depends critically on the choice of standard tree and we will defer its analysis to the next lemma.

Lemma 12 *For any vertex* $u \in S \setminus \{p_2, p_2'\}$ *we have* $\Delta_{\mathcal{T}}(u, q_2) < 3/2$.

Proof. We omit the proof in this short version. $\square$

We have now completed our analysis of standard trees. It remains to show that if a partition of $V = A \cup A'$ with $A \cap A' = \emptyset$ and $\sigma_A = \sigma'_A = 1/20$ exists, then we can choose a standard tree with dilation 3/2. We construct our standard tree $\mathcal{T}$ as follows: If $\alpha_i \in A$, then $\mathcal{T}$ contains $c_i d_i$ and $c'_i a'_{i+1}$, otherwise (that is, if $\alpha_i \in A'$) $\mathcal{T}$ contains $c'_i d'_i$ and $c_i a_{i+1}$.

Lemma 13 *The tree* $\mathcal{T}$ *constructed above has dilation* 3/2.

Proof. Lemmas 6 to 12 prove that we have $\Delta_{\mathcal{T}}(u, v) < 3/2$ for all pairs of points $u, v \in S$, except possibly for the pairs (d'_i, d_i) (with $1 \leqslant i \leqslant n$), (p_2, q_2), and (p'_2, q_2).

By construction, for any $1 \leqslant i \leqslant n$ either d_i is on the path from a_{n+1} to a_1, or d'_i is on the path from a'_{n+1} to a'_1. Hence $\Delta_{\mathcal{T}}(d'_i, d_i) < 3/2$ by Lemma 9.

It remains to check the dilation of (p_2, q_2) and (p'_2, q_2). We have:

$$\begin{aligned} d_{\mathcal{T}}(p_2, q_2) &= |p_2 a_{n+1}| + d_{\mathcal{T}}(a_{n+1}, a_1) + |a_1 q_1| + |q_1 q_2| \\ &= \tfrac{5}{9} 4^{n+1} - \tfrac{179}{90} + 5(4^n - 1) + \tfrac{1}{20} \\ &\quad + \tfrac{5}{2} + \tfrac{25}{9} 4^n - \tfrac{11}{18} = \tfrac{5}{2} 4^{n+1} - \tfrac{101}{20}. \end{aligned}$$

Since $|p_2 q_2| = \frac{5}{3} 4^{n+1} - \frac{101}{30}$, it follows that $\Delta_{\mathcal{T}}(p_2, q_2) = 3/2$. By a symmetric calculation we can show that $\Delta_{\mathcal{T}}(p'_2, q_2) = 3/2$. $\square$

3.4 Reduction on a Turing-machine

In the previous sections we have silently assumed that we can compute with real numbers. To complete our NP-hardness proof, we need to show that the reduction can be done on a Turing-machine or an equivalent model.

Recall that our input are n positive integers α_i. The input size is the total bit complexity of these n numbers. Now, we first observe that the coordinates of all points p_i, q_i, a_i, b_i, c_i are rational numbers and only depend on n. Their bit complexity is linear in n, and so they can be computed and represented exactly in polynomial time.

The difficulty are the points d_i, which are defined as the solution of a quadratic equation. We claim that it is sufficient to compute these points with a precision that can be realized with polynomial time and bit complexity.

Indeed, consider the SETPARTITION problem. If it has a positive solution, then there is a partition $V = A \cup A'$ with $\sigma_A = \sigma_{A'} = \sigma/2$. If it does not have a positive solution, then for any partition $V = A \cup A'$ we have $|\sigma_A - \sigma_{A'}| \geqslant 1$, simply because all numbers are integers. This means that we can give a lower bound strictly larger than 3/2 on the dilation of any spanning tree on S. This gap in the dilation between the case of a positive and negative answer allows us to replace the points d_i by an approximation that can be computed in polynomial time. We omit the details in this extended abstract.

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