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Spin-up for stratified fluid in a cylinder with time-dependent rotation rate

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Keywords : spin-up, Ekman number, Rossby number, stratification

Abstract

Numerical solutions for spin-up problem of a thermally stratified fluid in a cylinder with an insulating sidewall and time-dependent rotation rate are presented. Detailed results are given for aspect ratio of O(1), fixed Ekman number 10^{-4}, Rossby number 0.05 and Prandtl number O(1). Angular velocity of a cylinder wall changes with following formula, $\Omega_f = \Omega_i + \Delta\Omega[1-\exp(-t/t_c)]$. Here, this t_c value, which is very significant in present study, represents that how fast/slow the angular velocity of the cylinder wall reaches final angular velocity. The normalized azimuthal velocity and meridional flow plots for several tc value which cover ranges of the stratification parameter S(1 ~ 10) are presented. The role of viscous-diffusion and Coriolis term in present study is examined by diagnostic analysis of the azimuthal velocity equation.

1. Introduction

Spin-up refers to the general process of transient motion of a confined fluid by the external imposed change in the magnitude of the rotation rate of the circular container (radius R and height H). The closed container filled with a viscous incompressible fluid starts rotating about the longitudinal axis at rotation rate Ω_i. At the initial instant t=0, the rotation rate of the container is abruptly altered by an amount $\Delta\Omega$, and finally reaches a new rotation rate $\Omega_f = \Omega_i + \Delta\Omega$. This spin-up problem is a prominent issue in the transient dynamic of the rotating fluid system.

In the landmark paper, Greenspan & Howard (1963) examined the linearized analysis of a homogenous spin-up between two infinite disks for a small impulsive change in angular velocity of the disks and hence for a small Rossby number $\varepsilon \equiv \Delta\Omega / \Omega_i$. Also, they dealt with the case of the very small Ekman number $E \equiv \nu / \Omega_i H^2$, in which ν denotes the kinematic viscosity of fluid. This implies that the direct effect of viscosity is confined to the boundary layers adjacent to solid walls and is ignored in the interior inviscid core. This assumption is valid in most technological applications, such as turbomachinary, centrifuge and chemical mixers. Greenspan & Howard showed that the main dynamic ingredient is the fluid pumping mechanism by the Ekman layers which form on endwall disks (z=0, H) within time span of timescale $O(\Omega^{-1})$. This Ekman pumping leads to inward meridional secondary circulation. In the interior, the inward meridional flow motion brings forth adjustment of the new angular velocity by the angular momentum conservation and vortex-line stretching. This adjustment is substantially accomplished over the spin-up time scale $O(E^{-1/2}\Omega_i^{-1})$, rather than over the diffusion time scale $O(E^{-1}\Omega_i^{-1})$ which is order-of-magnitude bigger. Afterwards, this model have been demonstrated by the numerical and experimental endeavors (e. g., Warn-Varnas ea al. 1978).

The linearized, impulsive spin-up of a stratified fluid was analysed theoretically by Walin(1969) and Sakurai(1969). The structure of the Ekman layer near the endwall disks remains unchanged and the inward meridional circulation by Ekman pumping is still important dynamic element However, the meridional flow is limited to the interior area closer to Ekman layers because the vertical motion is inhibited by stratification. Therefore, the spin-up process is the spatial non-uniform and the azimutahl velocity near the Ekman layers converges faster than that in the area near the mid-height of the cylinder. And the overall rate of change in angular velocity of the interior is slower than for the homogenous spin-up. Hyun et al (1982) depicted the meridional flow and decay of azimuthal velocity in the numerical solutions. They showed that viscous-diffusion effect for stratified spin-up enhances in the interior and is the cause of the discrepancy with the theory.

Many research workers have only studied stratified spin-up case of the impulsive step change in angular velocity of the cylinder. In the present paper, the final angular velocity of the cylinder wall changes continuously with time-dependent formula, which is $\Omega_f = \Omega_i + \Delta\Omega[1-\exp(-t/t_c)]$. Here, t and t_c denote time and another variable. The t_c value is the significant parameter for present spin-up case and implies that how fast/slow the angular velocity of the walls reaches the final angular velocity Ω_f. The objective of this paper is to investigates differences of the dynamic characteristics between a usual stratified spin-up and present stratified spin-up with time-dependent rotation rate.

2. Formulation and numerical method

Consider a vertically-mounted closed cylindrical container (radius R and height H) that is completely filled with incompressible Boussinesq fluid having kinematic viscosity ν, thermometric diffusivity κ and coefficient of volumetric expansion α. These physical properties of the fluid are assumed to be constant. Also, the aspect ratio (=R/H) is fixed O(1). The top and bottom disks are kept at constant temperature to produce a stable stratification by $\Delta T (T_T - T_B > O)$ over H. The temperatures of the top and bottom disk are T_T and T_B, respectively. The Boussinesq approximation, which is $\rho = \rho_B[1-\alpha(T-T_B)]$, is invoked to express the variation of density ρ. Here, ρ_B and T_B are reference density and temperature of the bottom disk. Then the walls of cylinder suddenly rotate abovementioned final angular velocity Ω_f. This is well expressed in a Fig. 1.

We employed cylindrical coordinates (r, φ, z) with corresponding velocity component (u, v, w), viewed in the rotating frame. These

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non-dimensional equations are well documented. (e.g., Hyun et al, 1982). The nondimensional temperature θ is defined as $\theta \equiv \frac{T - T_B}{T_T - T_B}$. The Prandtl number Pr $\left[\equiv \frac{\nu}{\kappa} \right]$ is assumed O(1) and the stratification number S $\left[\equiv \frac{\alpha g (T_T - T_B)}{H \Omega_i^2} \right]$ represents the overall buoyancy effect. The numerical solution technique of equations is a finite volume procedure, which is based on the well-established SIMPLER algorithm (Patankar, 1980). And the QUICK scheme (Hayase et al, 1994) was employed to deal with the nonlinear advection terms. Time step was based on an iterative Eulerian implicit method of accuracy O(Δt). Grid-stretching was implemented to resolve thin boundary layers near walls. Convergence of the solutions was declared at each time step when the maximum relative change between consecutive iteration levels fell below 10^{-4} for u, v, w, θ.

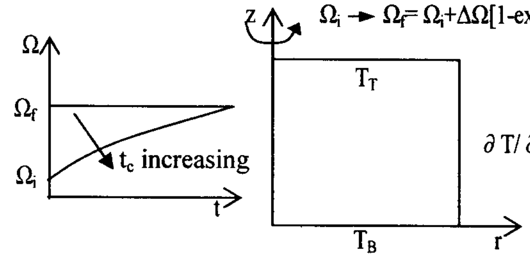

Fig. 1 Schematic diagram

3. Results

For $E=10^{-4}$, Ekman layer forming time scale, spin-up time scale and diffusion time scale are 1, 10^2 and 10^4, respectively. So these values were applied as t_c parameter except for using $t_c=10^3$ instead of diffusion time scale $t_c=10^4$, in which case flow motion is extremely very weak and unrealistic. The flow dynamic patterns of $t_c=10^3$ case is same those of $t_c=10^4$ case, only there are differences of absolute magnitude of flow value.

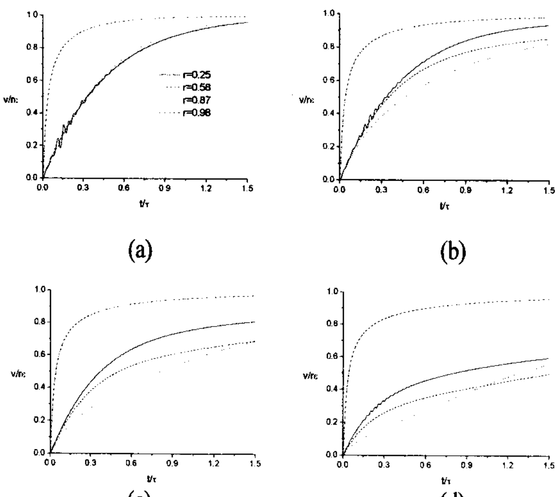

Fig. 2 Normalized azimuthal velocity at z=0.25(t_c=0)
(a) S=0; (b) S=1; (c) S=3; (a) S=8

Fig.2 displays step change of rotation rate like classical spin-up problem (t_c=0) and the azimuthal velocity of Fig. 2 is scaled by rε. Fig, 2(a) shows the results for a homogenous spin-up. The scaled azimuthal flow is uniform in the interior, and this implies solid body rotation of the interior. And the azimuthal velocity at sidewall viscous boundary layer is spun up much more rapidly. Fig. 2(b, c, d) are results of stratified spin-up problem. The scaled azimuthal velocity is no longer uniform in the interior. The azimuthal velocity at sidewall viscous region converges faster than that at the interior, in which pattern is equal to homogenous fluid case. However, in the interior far from sidewall, spin-up proceeds initially more rapidly at smaller radii and decreases as radial position increases. This phenomenon is because the corner jet exists for stratified fluid case. This implies that vortex-stretching by secondary circulation is stronger at smaller radii than at the large radius. Also, the results of Fig. 2(b, c, d), in which stratification numbers are 1, 3, and 8, respectively, show that the spin-up process in the interior is much weaker as the stratification increases.

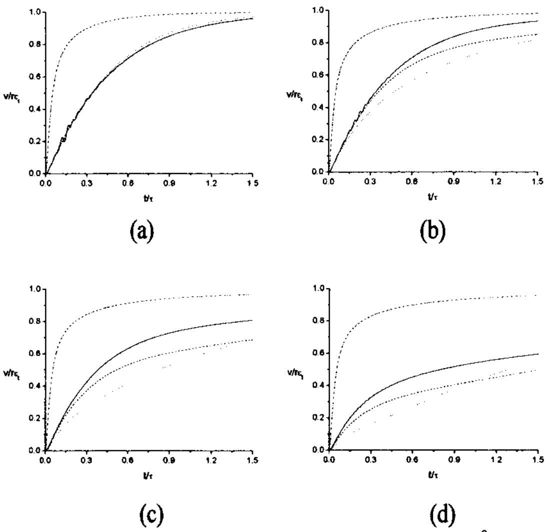

Fig. 3 Same as in Fig. 2, except for $t_c=E^0$

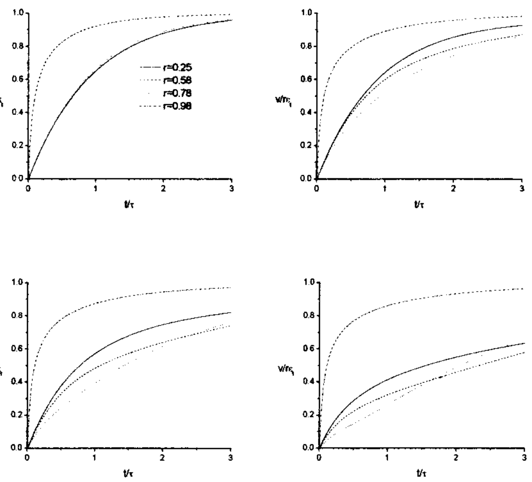

Fig. 4 Same as in Fig. 2, except for $t_c=E^{-1/2}$

The plots of Fig. 3-5, in turn $t_c=E^0$, $E^{-1/2}$ and $0.1E^{-1}$, are results of spin-up with time-dependent rotation rate. The azimuthal velocity is normalized by rε_t. This tells us velocity information relative to velocity of rotating cylinder wall. The formula of ε_t is $\varepsilon_t = \varepsilon[1-\exp(-t/t_c)]$, which is not constant parameter and implies a realistic increase increment of angular velocity of walls at each moment. The results of t_c=1 are nearly identical with step change case, so the convergence time of fluid is nearly same. As the t_c value increases, shown plots Fig.4 and 5, the convergence of the azimuthal velocity by scaled rε_t is late somewhat at given time and the overall adjustment of fluid takes a lat of time more than O(τ). But, in general, the effect of stratification and the convergence pattern of fluid are applied without discrimination as the case of step change spin-up regardless of t_c value.

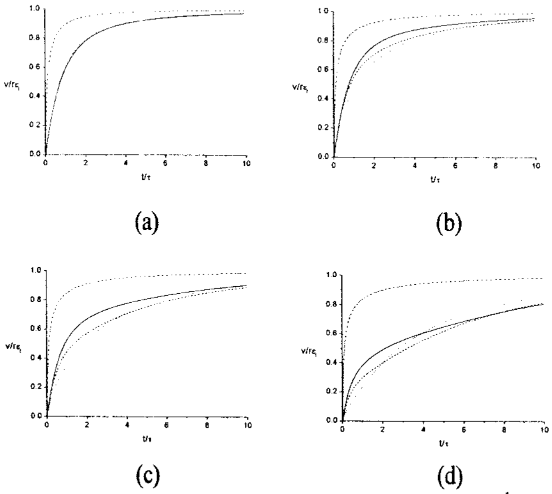

Fig. 5 Same as in Fig. 2, except for t_c=0.1E^{-1}

Note that the dominant flow is in the azimuthal direction. Therefore, diagnostic studies of the azimuthal velocity equation were performed to examine physical dynamic effect of the present spin-up problem. The form of azimuthal velocity equation is of primary interest:

$$\frac{\partial v}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial r}(ruv)-\frac{\partial}{\partial z}(vw)-\frac{vu}{r}-2u+E(\nabla^2 v-\frac{v}{r^2}) \quad . \qquad (1)$$

The left-hand-side term represents the time-dependent part, or azimuthal acceleration. The first four terms on right-hand-side of the form (1) stand for the inviscid dynamic effects. They are radial advection, vertical advection, curvature term and Coriolis acceleration, respectively. The fifth term denotes viscous-diffusion effect. The nonlinear advection terms and curvature term are several orders of magnitude smaller than Coriolis term for small Rossby number considered in present problem. Fig. 6 shows plots of evolution of Coriolis and viscous-diffusion terms at two vertical positions on mid-radius for some stratification numbers and three t_c values. These results show that the Coriolis term in the interior away from mid-height is generally larger than at mid-height for every case. For t_c=0.1E^{-1} case, the viscous-diffusion term plays a principal role even before the fluid doesn't reach at state of final angular velocity. The stratification effect reinforces this phenomenon. The viscous term at mid-depth is lager than viscous term at position away from mid-depth, and the differences of two viscous values are bigger after spin-up time scale O(τ). These facts tell us that the viscous-diffusion term can not be negligible in the case of spin-up with slowly rotating cylinder.

The radial profiles of viscous-diffusion term during various temporal stages near mid-depth are displayed in Fig. 7. For the homogenous fluid, the viscous dynamic effect is only limited to the sidewall boundary layer in the every case. But the viscous dynamic effect is not restricted within sidewall boundary layer and penetrates in the interior region as the t_c parameter and stratification number increase. For example, in the case of t_c=0.1E^{-1} and S=8, this dynamic term affects nearly even near central axis. These results support the results of Fig. 6 plots.

Fig. 8 shows contour plots of meridional stream function ψ, which is defined as $u=\frac{1}{r}\frac{\partial\Psi}{\partial z}$ and $w=-\frac{1}{r}\frac{\partial\Psi}{\partial r}$. Dotted (solid) lines represent counter-clockwise (clockwise) circulations in Ψ plots. This arises by Ekman pumping effect of fluid toward the disk in the central axial region. The left plots of Fig. 8 are step change spin-up like classical problem. As is evident, meridional flow is less intense in middle region as the stratification increases. This reflects in an axially non-uniform rate of spin-up. The cases of several t_c value maintain these results like classical spin-up, though there are differences of meridional flow strength. As t_c value increase, the variation of the meridional circulation is reduced despite the increase of stratification effect.

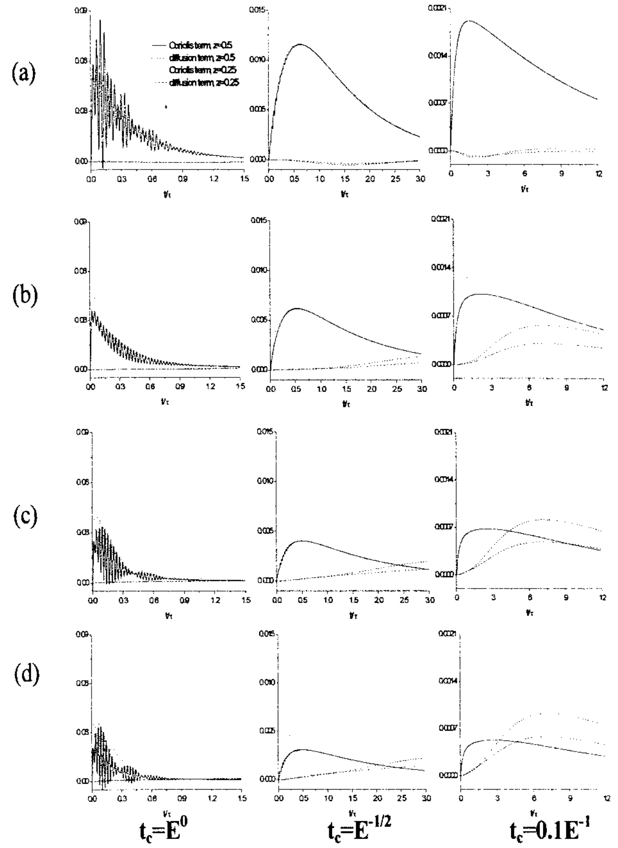

Fig. 6 Evolutions of the Coriolis term and diffusion term in the mid-radius
(a) S=1; (b) S=5; (c) S=8; (d) S=10

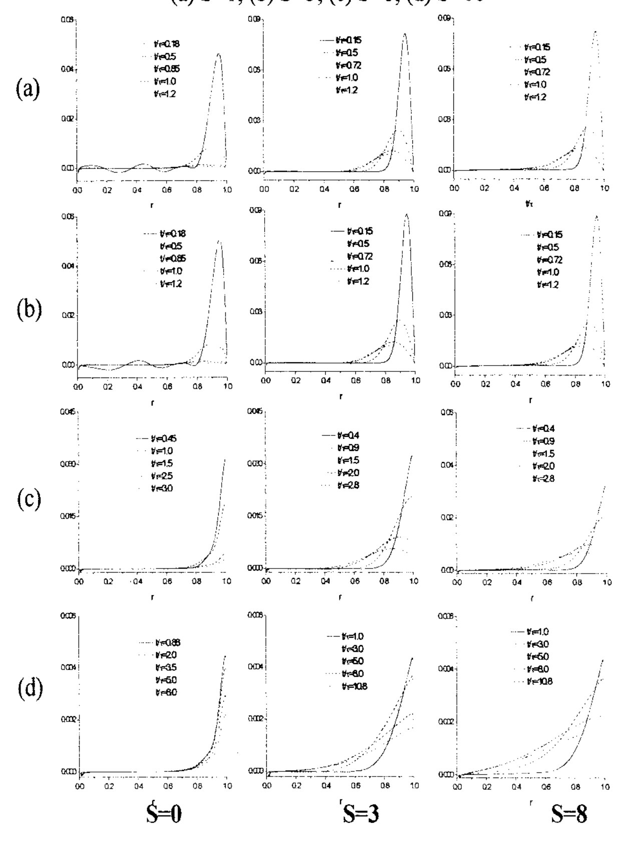

Fig. 7 The radial plots of diffusion term at several times (z=0.5)
(a) t_c=0; (b) t_c=E^0; (c) t_c=$E^{-1/2}$; (b) t_c=0.1E^{-1};

4. Conclusion

Extensive numerical solutions to unsteady incompressible Navier-Stokes equations have been examined for stratified spin-up with time-dependent rotation rate and insulating sidewall.

The effect of stratification and the convergence pattern of fluid are analogous to the case of step change spin-up regardless of t_c value, only there are the differences of absolute magnitude of each values and adjustment time of fluid. But the viscous dynamic effect is not limited within sidewall boundary layer and penetrates in the interior region as the t_c parameter increases, and the viscous term at mid-depth is lager than viscous term at position away from mid-depth, and the differences of two viscous values are bigger after spin-up time scale O(τ). The increase of the stratification number accelerates these phenomena. That the meridional flow is confined close to Ekman layer for stratified fluid is not changed for large t_c parameter. As the t_c value increases, the stratification effect in meridional flow is reduced.

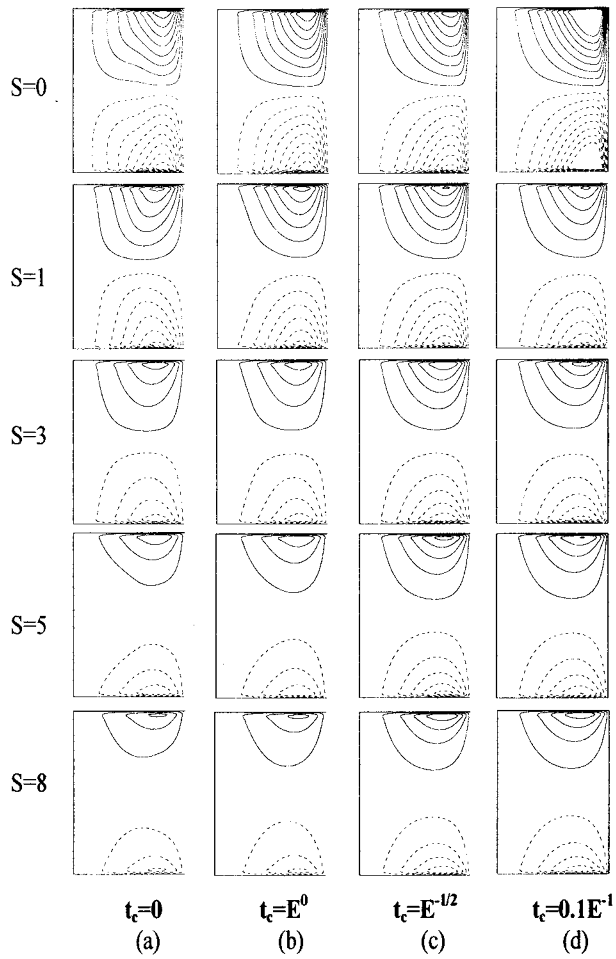

Fig. 8 Sequential plots of meridional stream function in the r-z plane
(a) t/τ=0.6, Δψ=4.83×10^{-6}; (b) t/τ=0.6, Δψ=4.83×10^{-6}
(c) t/τ=2.0, Δψ=1.89×10^{-6}; (d) t/τ=10.0, Δψ=4.13×10^{-7}

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