2005 한국경영과학회/대한산업공학회 춘계공동학술대회
2005 년 5 월 13 일~14 일, 충북대학교

A simple computational procedure to obtain the queue-length distribution of the discrete-time GI/G/1 queue

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Abstract

Based on a discrete-time version of the distributional Little's law, we present a simple computational procedure to obtain the queue-length distribution of the discrete-time GI/G/1 queue from its waiting-time distribution that is available by various existing methods. We also discuss our numerical experience and address a couple of remarks on possible extensions of the procedure.

1. Introduction

Discrete-time queueing models have been given a growing attention due to their applications to a variety of slotted digital communication systems and other related areas. In discrete-time queues, the time axis is segmented into a sequence of equal intervals, called *slot*s, and arrivals and departures of customers are assumed to take place at slot boundaries. In this paper, we consider the discrete-time single-server GI/G/1 queue, where both the interarrival and service times are sequences of independent and identically distributed (i.i.d.) general discrete random variables that are independent of each other.

While the continuous-time GI/G/1 queue is difficult to analyze both mathematically as well as numerically, it is interesting to note that its discrete-time counterpart is much easier to do so. In this paper, we present a simple computational procedure to obtain the stationary queue-length distribution from its stationary waiting-time distribution by means of a discrete-time version of the so-called *distributional Little's law* (its continuous-time version is established by Haji and Newell 1971).

Recent advances in the analysis of the discrete-time GI/G/1 queue-length distribution include the following. Haβlinger (1995) shows that the stationary queue-length distributions of the finite- as well as infinite-capacity discrete-time GI/G/1 queue can be represented in terms of the so-called *characteristic zero*s. Yang and Chaudhry (1996) use the *matrix-analytic method* (MAM) to analyze the arrival- and departure-time embedded Markov chains arising in this queue, which turn out to be of the GI/M/1 and M/G/1 types, respectively. Alfa and Li (2001) show that this queue can be easily set up as a *quasi-birth-death* *process* and use MAM to obtain the stationary distributions of the random variables of interest, such as the queue length, the waiting time, and the length of a busy period. Alfa (2003) also extends this result to the batch-arrival $GI^X/G/1$ queue.

In this paper, we establish in Section 2 a discrete-time version of the distributional Little's law that relates the stationary queue-length distribution to the stationary waiting-time distribution. Using this relation, we show in Section 3 that one can obtain the queue-length distribution of the discrete-time GI/G/1 queue from its waiting-time distribution that is available by various existing methods. Finally, we discuss our numerical experience and address a couple of remarks on possible extensions of the computational procedure.

2. Distributional Little's Law

In this section, we establish a discrete-time version of the so-called distributional Little's law that relates the stationary number of customers in system at an arbitrary time (that falls somewhere in the middle of a slot with probability 1) to the stationary number of slots a customer spends in system.

Although the distributional Little's law applies to a broad class of queueing system, we refine ourselves in this paper to a stationary discrete-time FIFO (First In First Out) GI/G/1 queue, where A_n is the interarrival time between customers C_n and C_{n+1}, S_n is the service time of C_n, and $\{A_n\}$ and $\{S_n\}$ are independent sequences of i.i.d. general discrete positive random variables. Let interarrival and service times be denoted by generic random variables A and S with their respective *probability generating function*s (PGFs) $A(z)$ and $S(z)$. We assume $\rho = E(S)/E(A) < 1$ to ensure stability.

Consider the number of elapsed slots since the last arrival, which is denoted by A_E with its PGF $A_E(z) = (1 - A(z))/(E(A)(1-z))$. Also consider a discrete-time *equilibrium renewal arrival process*, where the number of slots up to the first renewal arrival is '$A_E + 1$.' Then, to the discrete-time GI/G/1 queue, we apply the same arguments as is presented to establish the continuous-time distributional Little's law (Haji and Newell 1971). As a result, we have

$$P(N = n) = P(\Lambda(W) = n), \quad n = 0,1,2,\ldots, \qquad (1)$$

where N is the stationary queue length, i.e., the number of customers in system (by system, we mean queue plus server), W is the stationary waiting time, i.e., the number of slots a customer spends in system, and $\Lambda(i)$ is the number of renewal arrivals during $(0,i]$, $i = 1,2,\ldots$, with $\Lambda(0) = 0$, in the discrete-time equilibrium renewal arrival process that is independent of W.

3. Computational Procedure

Making use of relation (1), we present in this section a computational procedure to get the queue-length distribution of the discrete-time GI/G/1 queue from its waiting-time distribution.

Since the distribution of W is easily available, e.g., from the iterative method based on the Wiener-Hopf factorization (Grassmann and Jain 1989), the solution presented in terms of zeros outside the unit circle (Chaudhry 1993), or MAM (Alfa and Li 2001), the calculation of the queue-length probability through (1) is now reduced to counting the number of renewal arrivals during $(0,i]$ for a given $W=i$:

$$P(N=n)=\sum_{i=0}^{\infty}P(\Lambda(i)=n)P(W=i). \qquad (2)$$

To do this, let $P_n(z)$ be the *generating function* (GF) of $P(\Lambda(i)=n)$ for $i=1,2,3,\cdots$; i.e., $P_n(z)=P(\Lambda(1)=n)z^1+P(\Lambda(2)=n)z^2+\cdots$. Then it is given by

$$P_n(z)=\frac{zA_E(z)A(z)^{n-1}\left(1-A(z)\right)}{1-z},\quad n=1,2,\cdots, \qquad (3)$$

$$P_0(z)=\frac{z\left(1-A_E(z)\right)}{1-z}. \qquad (4)$$

As illustrated by Kim and Chaudhry (2005), for any given $n=0,1,2,\cdots$, (3) and (4) are easily expanded into power series to give all the values of $P(\Lambda(i)=n)$ of interest with $i=1,2,3,\cdots$. Substituting these values into (2), one can calculate a complete queue-length distribution.

4. Numerical Experience and Some Remarks

In this section, we discuss our numerical experience and address a couple of remarks on possible extensions of the procedure.

We tested our computational procedure using the same GI/G/1 queue as is considered by Alfa (2003) and confirmed that our results nicely agree with his. No problems have been encountered in applying the procedure to a variety of the GI/G/1 queues. (Sample numerical results are demonstrated at the talk.)

Finally, we remark that the procedure presented here can also be applied to the other discrete-time queues in which the discrete-time distributional Little's law is still valid. Examples include the GI/G/1 queue with multiple vacations and the multi-server GI/D/c queue with deterministic service times, in which it can be shown that relation (1) still holds. Along the same lines as presented in this paper, their stationary queue-length distributions can be obtained from their respective waiting-time distributions that seem to be easily achievable.

References

A.S. Alfa, Combined elapsed time and matrix-analytic method for the discrete-time GI/G/1 and GIX/G/1 systems, Queueing. Sys. 45 (2003) 5–25.

A.S. Alfa, W. Li, Matrix-geometric analysis of the discrete time GI/G/1 system, Stoch. Models. 17 (2001) 541–554.

M.L. Chaudhry, Alternative numerical solutions of stationary queueing-time distributions in discrete-time queues: GI/G/1, J. Opl. Res. Soc. 44 (1993) 1035–1051.

W.K. Grassmann, J.L. Jain, Numerical solutions of the waiting time distribution and idle time distribution of the arithmetic GI/G/1 queue, Oper. Res. 37 (1989) 141–150.

R. Haji, G.F. Newell, A relation between stationary queue and waiting time distributions, J. Appl. Probab. 8 (1971) 617–620.

G. Haβlinger, A polynomial factorization approach to the discrete-time GI/G/1/(N) queue size distribution, Perfrom. Eval. 23 (1995) 217–240.

N.K. Kim, M.L. Chaudhry, Numerical inversion of generating functions - a computational experience, working paper. 2005.

T. Yang, M.L. Chaudhry, On steady-state queue size distributions of the discrete-time GI/G/1 queue, Adv. Appl. Probab. 28 (1996) 1177-1200.