Receding Horizon H_∞ Predictive Control for Linear State-delay Systems

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Abstract: This paper proposes the receding horizon H_∞ predictive control (RHHPC) for systems with a state-delay. We first proposes a new cost function for a finite horizon dynamic game problem. The proposed cost function includes two terminal weighting terms, each of which is parameterized by a positive definite matrix, called a terminal weighting matrix. Secondly, we derive the RHHPC from the solution to the finite dynamic game problem. Thirdly, we propose an LMI condition under which the saddle point value satisfies the well-known nonincreasing monotonicity. Finally, we shows the asymptotic stability and H_∞-norm boundedness of the closed-loop system controlled by the proposed RHHPC. Through a numerical example, we show that the proposed RHHC is stabilizing and satisfies the infinite horizon H_∞-norm bound.

Keywords: Receding horizon H_∞ control, Terminal weighting matrix, Nonincreasing monotonicity, H_∞-norm bound.

1. Introduction

Receding horizon controls (RHC) have attracted much attention from academia and industry because of its ability to handle time-varying systems, disturbances, input constraint, time-varying tracking commands, and nonlinear systems. Most research results about the RHC are concentrated on ordinary systems without delays. However, RHC approaches for systems with a state-delay can be found only recently [1], [2]. In [1], a simple control method based on the receding horizon concept is proposed. However, it does not have a state weighting in the cost function. Furthermore, it does not guarantee closed-loop stability at the stage when RHC solution is obtained. The stability can be checked only after the RHC solution is obtained. General extension of the RHC to state-delayed systems is proposed in [2].

The RHC for ordinary systems has been applied to H_∞ problems recently in order to combine the practical advantage of the RHC with the robustness of the H_∞ control [3]-[6]. The saddle point value in H_∞ problems corresponds to the optimal cost in LQ problems. Those results in [3]-[6] propose conditions for nonincreasing monotonicity of the saddle point value. For state-delayed systems, there have been many approaches for H_∞ problems [7], [8], [9]. Among them, the results in [8] consider a finite horizon H_∞ control problems. However, since it deals with finite horizon problem, the closed-loop stability issue was not dealt with.

In this paper, we make the first approach to extend the receding horizon H_∞ predictive control (RHHPC) to state-delayed systems. The proposed RHHPC has asymptotic closed-loop stability and satisfies the infinite horizon H_∞-norm bound. We will propose a condition for nonincreasing monotonicity of the optimal cost, which plays an important role for stability and H_∞-norm boundedness.

2. Receding Horizon H_∞ Control

Consider a linear time-invariant system with a state-delay

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + Bu(t) + B_w w(t) \quad (1)$$
$$z(t) = Cx(t) + Du(t) \quad (2)$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ the control, $w \in \mathbf{R}^l$ the disturbance, $z \in \mathbf{R}^p$ the controlled output, $h > 0$ the delay, respectively. It is assumed that $C^TD = 0$ and $D^TD = I$. In order to obtain an RHHPC, we will first consider the following cost function with the finite horizon

$$\begin{aligned} & J(x_{t_0}, t_0, t_f, u, w) \\ = & \int_{t_0}^{t_f} [x^T(\tau)Qx(\tau) + u^T(\tau)u(\tau) - \gamma^2 w^T(\tau)w(\tau)]d\tau \\ & + x^T(t_f)F_1x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)F_2x(\tau)d\tau, \end{aligned}$$

where $Q = C^TC > 0$, $F_1 > 0, F_2 > 0$. We can regard J as a function of either $\mathcal{L}_2$ signals of feedback strategies. Let $\mathcal{M}$ $=\{\mu : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}^m\}$ and $\mathcal{N}$ $=\{\nu : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}^l\}$, where $\mathbf{C_n[-h, 0]}$ is the space of n-dimensional vector functions continuous on $[-h, 0]$. Spaces $\mathcal{M}$ and $\mathcal{N}$ are the strategy spaces, and we will write strategies as μ, ν to distinguish them from signals u, w.
Denote $x_t = x(t+\theta)$, $h \in [-h, 0]$. $x_t \in \mathbf{C}_n[-h, 0]$ by the definition of $\mathbf{C}_n[-h, 0]$. Let's formulate a dynamic game problem

$$\min_{\mu \in \mathcal{M}} \max_{\nu \in \mathcal{N}} J(x_{t_0}, t_0, t_f, \mu, \nu), \quad (3)$$

which is a zero sum game, where u is the minimizing player and w is the maximizing player. The optimal u and the worst case w are called saddle point strategies. A saddle point solution $u(\tau) = \mu^*(\tau, x(\tau))$, $w(t) = \nu^*(\tau, x(\tau))$ satisfies

$$J(x_{t_0}, t_0, t_f, \mu^*, w) \le J(x_{t_0}, t_0, t_f, \mu^*, \nu^*) \le J(x_{t_0}, t_0, t_f, u, \nu^*)$$

for all $u, w \in \mathcal{L}_2[t_0, t_f]$. For simple notation, $J(x_{t_0}, t_0, t_f, \mu^*, \nu^*)$ will be denoted by $J^*(x_{t_0}, t_0, t_f)$ throughout this paper. The purpose of this paper is to develop a method to design a control law, u_R, based on the receding horizon concept such that

(a) in case of zero disturbance, the closed-loop system is asymptotically stable
(b) with zero initial condition, the closed-loop transfer function T_{zw} satisfies the H_∞-norm bound, for given $\gamma > 0$,

$$\|T_{zw}\|_\infty \le \gamma. \quad (4)$$

Since the control is based on the receding horizon strategy and the closed-loop system satisfies the H_∞-norm bound, we will call it receding horizon H_∞ predictive control (RHHPC) throughout this paper.

Remark 1: It is noted that the terminal weighting function consists of two terms, parameterized by two matrices F_1 and F_2. We will call them terminal weighting matrices in this paper. The purpose of adding a second terminal weighting term, parameterized by F_2, is to take the delay effect into account such that the resulting RHHPC is stabilizing. More specifically, if F_2 is chosen properly, the saddle point value satisfy the well-known 'nonincreasing monotonicity property', which will be considered in Section 3.

Before moving on, we introduce a lemma, which gives the sufficient condition for the existence of a saddle point solution. In the lemma, $V(\tau, x_\tau) : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}$ denotes a continuous and differentiable functional. Furthermore, we will use the notation

$$\frac{d}{d\tau}V(\tau,x_\tau)\Big|_{\substack{\mu(\tau,x_\tau)\\ \nu(\tau,x_\tau)}} \triangleq \lim_{\Delta\tau\to 0}\left[\frac{V(\tau+\Delta\tau, x^{\mu,\nu}_{\tau+\Delta\tau}) - V(\tau,x_\tau)}{\Delta\tau}\right]$$

where $x^{\mu,\nu}_{\tau+\Delta\tau} = x(\tau+\Delta\tau+s),\ s\in[-h,0]$ is the solution of the system (1) resulting from the control $u(t) = \mu(t, x_\tau)$ and disturbance $w(t) = \nu(t, x_\tau)$.

Lemma 1: If it is possible to find a continuous functional $V(\tau, x_\tau) : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}$, and a vector functional $\mu^*(\tau, x_\tau) : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}^m$ and $\nu^*(\tau, x_\tau) : [t_0, t_f] \times \mathbf{C}_n[-h, 0] \to \mathbf{R}^1$ such that

$$(a)\quad V(t_f, x_{t_f}) = x^T(t_f)F_1x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)F_2x(\tau)d\tau$$

$$\begin{aligned}(b)\quad & \frac{d}{d\tau}V(\tau,x_\tau)\Big|_{\substack{\mu^*(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)Qx(\tau)\\ & +\mu^{*T}(\tau,x_\tau)\mu^*(\tau,x_\tau) - \gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau) = 0\end{aligned}$$

$$\begin{aligned}(c)\quad & \frac{d}{d\tau}V(\tau,x_\tau)\Big|_{\substack{\mu^*(\tau,x_\tau)\\ \nu(\tau,x_\tau)}} + x^T(\tau)Qx(\tau)\\ & +\mu^{*T}(\tau,x_\tau)\mu^*(\tau,x_\tau) - \gamma^2\nu^{T}(\tau,x_\tau)\nu(\tau,x_\tau)\\ & \le \frac{d}{d\tau}V(\tau,x_\tau)\Big|_{\substack{\mu^*(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)Qx(\tau)\\ & +\mu^{*T}(\tau,x_\tau)\mu^*(\tau,x_\tau) - \gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau)\\ & \le \frac{d}{d\tau}V(\tau,x_\tau)\Big|_{\substack{\mu(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)Qx(\tau)\\ & +\mu^{T}(\tau,x_\tau)\mu(\tau,x_\tau) - \gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau)\end{aligned}$$

for all $\tau \in [t_0, t_f]$ and all $x_\tau \in \mathbf{C}_n[-h, 0]$. Then, $V(s, x_s) = J(x_s, s, t_f, \mu^*, \nu^*)$ and

$$J(x_s,s,t_f,\mu^*,\nu) \le J(x_s,s,t_f,\mu^*,\nu^*) \le J(x_s,s,t_f,\mu,\nu^*) \tag{5}$$

for all $s \in [t_0, t_f]$. That is, $u(\tau) = \mu^*(\tau, x_\tau)$ and $w(\tau) = \nu^*(\tau, x_\tau)$ are saddle point solutions and $V(\tau, x_\tau)$ is a saddle point value.

Proof: Integrating (b) from s to t_f yields

$$\begin{aligned}V(t_f,x_{t_f}) - V(s,x_s) + \int_s^{t_f}[x^T(\tau)Qx(\tau)\\ +\mu^{*T}(\tau,x_\tau)\mu^*(\tau,x_\tau) - \gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau)]d\tau = 0\end{aligned}$$

From (a),

$$\begin{aligned}V(s,x_s) &= \int_s^{t_f}[x^T(\tau)Qx(\tau) + \mu^{*T}(\tau,x_\tau)\mu^*(\tau,x_\tau)\\ &\quad -\gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau)]d\tau\\ &\quad +x^T(t_f)F_1x(t_f) + \int_{t_f-h}^{t_f} x^T(\tau)F_2x(\tau)d\tau\\ &= J(x_s,s,t_f,\mu^*,\nu^*)\end{aligned}$$

From (c),

$$\begin{aligned}&\frac{d}{d\tau}V(\tau,x_\tau)\Big|_{\substack{\mu(\tau,x_\tau)\\ \nu^*(\tau,x_\tau)}} + x^T(\tau)Qx(\tau) + \mu^T(\tau,x_\tau)\mu(\tau,x_\tau)\\ &-\gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau) \ge 0\end{aligned}$$

Integrating the above from s to t_f yields

$$\begin{aligned}V(s,x_s) &\le \int_{t_0}^{t_f}[x^T(\tau)Qx(\tau) + \mu^{T}(\tau,x_\tau)\mu(\tau,x_\tau)\\ &\quad -\gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau)]d\tau + x^T(t_f)F_1x(t_f)\\ &\quad + \int_{t_f-h}^{t_f} x^T(\tau)F_2x(\tau)d\tau\\ &= J(x_s,s,t_f,\mu,\nu^*)\end{aligned}$$

Similarly, we have

$$V(s,x_s) \ge J(x_s,s,t_f,\mu^*,\nu)$$

If we take $s = t_0$, then

$$\begin{aligned}J(x_{t_0},t_0,t_f,\mu^*,\nu) &\le V(t_0,x_{t_0}) = J(x_{t_0},t_0,t_f,\mu^*,\nu^*)\\ &\le J(x_{t_0},t_0,t_f,\mu,\nu^*).\end{aligned}$$

This completes the proof. ∎

From the above lemma, we see that $V(\tau, x_\tau)$ is a saddle point value, that is, $V(\tau, x_\tau) = J^*(x_\tau, \tau, t_f)$. Furthermore, it is noted that $V(s, x_s) \ge 0$ for all $s \in [t_0, t_f]$. This can be verified from Lemma 1.

Before deriving the receding horizon predictive control, we first provide a solution to a finite horizon dynamic game problem. The derivation is based on the above lemma. The procedure taken for derivation of the solution is similar to that used in [2]. Readers are referred to [2]. We assume the saddle point value has the form

$$V(\tau,x_\tau) = \tag{6}$$
$$\begin{cases} x^T(\tau)P_1(\tau)x(\tau) + 2x^T(\tau)\int_{-h}^0 P_2(\tau,s)x(\tau+s)ds\\ +\int_{-h}^0\int_{-h}^0 x^T(\tau+r)P_3(\tau,r,s)x(\tau+s)drds\\ \text{for } t_0 \le \tau < t_f - h\\ \\ x^T(\tau)W_1(\tau)x(\tau)\\ +2x^T(\tau)\int_{-h}^{\tau_-\tau-h} W_2(\tau,s)x(\tau+s)ds\\ +\int_{-h}^{t_f-t-h}\int_{-h}^{t_f-t-h} x^T(\tau+r)W_3(\tau,r,s)x(\tau+s)drds\\ +\int_{t_f-h}^{\tau} x^T(\tau)F_2x(\tau)d\tau \text{ for } t_f-h \le \tau \le t_f.\end{cases}$$

Using the above saddle point value, the saddle point strategies for the dynamic game problem in (3) are given by

$$\mu^*(\tau, x_\tau) = \begin{cases} -B^T\left[P_1(\tau)x(\tau) + \int_{-h}^{0} P_2(\tau,s)x(\tau+s)ds\right] \\ \text{for } t_0 \le \tau < t_f - h. \\ -B^T\left[W_1(\tau)x(\tau) + \int_{-h}^{t_f-t-h} W_2(\tau,s)x(\tau+s)ds\right] \\ \text{for } t_f - h \le \tau \le t_f \end{cases}$$

and

$$\nu^*(\tau, x_\tau) = \begin{cases} \gamma^{-2}B_w^T\left[P_1(\tau)x(\tau) + \int_{-h}^{0} P_2(\tau,s)x(\tau+s)ds\right] \\ \text{for } t_0 \le \tau < t_f - h. \\ \gamma^{-2}B_w^T\left[W_1(\tau)x(\tau) + \int_{-h}^{t_f-t-h} W_2(\tau,s)x(\tau+s)ds\right] \\ \text{for } t_f - h \le \tau \le t_f. \end{cases}$$

P_1, P_2, and P_3 satisfy the following Riccati-type partial differential equations:

$$\begin{aligned} \dot{P}_1(\tau) + A^T P_1(\tau) + P_1(\tau)A & \\ -P_1(\tau)(BB^T - \gamma^{-2}B_wB_w^T)P_1(\tau) & \\ +Q + P_2(\tau,0) + P_2^T(\tau,0) &= 0 \quad (7) \\ \left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial s}\right)P_2(\tau,s) + A^T P_2(\tau,s) + P_3(\tau,0,s) & \\ -P_1(\tau)(BB^T - \gamma^{-2}B_wB_w^T)P_2(\tau,s) &= 0 \quad (8) \\ \left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)P_3(\tau,r,s) & \\ -P_2^T(\tau,r)(BB^T - \gamma^{-2}B_wB_w^T)P_2(\tau,s) &= 0 \quad (9) \end{aligned}$$

with boundary conditions

$$\begin{aligned} P_2(\tau,-h) &= P_1(\tau)A_1 & (10) \\ P_3(\tau,-h,s) &= A_1^T P_2(\tau,s), & (11) \end{aligned}$$

where $t_0 \le \tau < t_f - h$, $-h \le r \le 0$ and $-h \le s \le 0$. Furthermore, W_1, W_2, and W_3 satisfy the following Riccati-type partial differential equations:

$$\begin{aligned} \dot{W}_1(\tau) + A^T W_1(\tau) + W_1(\tau)A & \\ -W_1(\tau)(BB^T - \gamma^{-2}B_w{B_w}^T)W_1(\tau) + Q + F_2 &= 0 \\ \left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial s}\right)W_2(\tau,s) + A^T W_2(\tau,s) & \\ -W_1(\tau)(BB^T - \gamma^{-2}B_wB_w^T)W_2(\tau,s) &= 0 \\ \left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)W_3(\tau,r,s) & \\ -W_2^T(\tau,r)(BB^T - \gamma^{-2}B_wB_w^T)W_2(\tau,s) &= 0 \end{aligned}$$

with boundary condition

$$\begin{aligned} W_2(\tau,-h) &= W_1(\tau)A_1 \\ W_3(\tau,-h,s) &= A_1^T W_2(\tau,s), \end{aligned}$$

where $t_f - h \le \tau < t_f$, $-h \le r \le 0$ and $-h \le s \le 0$. In addition, P_i and W_i satisfy the following boundary conditions:

$$\begin{aligned} W_1(t_f) &= F_1 & (12) \\ P_1(t_f - h) &= W_1(t_f - h) & (13) \\ P_2(t_f - h, s) &= W_2(t_f - h, s) & (14) \\ P_3(t_f - h, r, s) &= W_3(t_f - h, r, s). & (15) \end{aligned}$$

P_1, P_2, P_3 and W_1, W_2, W_3 are solved backward in time from t_f to t_0. Because the system is time-invariant, the shape of P_i and W_i are only characterized by the difference between the initial time and the final time, that is, $t_f - t_0$. If $t_f - t_0$ varies, the values of P_i and W_i at the initial time, t_0, also vary. However, if $t_f - t_0$ is fixed to a constant value, the values are all the same at initial time. For example, $P_1(t_0)$ with $t_0 = 1$ and $t_f = 5$ is equal to $P_1(t_0)$ with $t_0 = 2$ and $t_f = 6$. If we take receding horizon strategy, t_0 and t_f corresponds to t, current time, and $t + T_p$, respectively. Therefore, the difference between the initial time and the terminal time is always constant as T_p. Therefore, $P_1(t_0)$ reduces to a constant matrix regardless of the value of t_0. Let's introduce new notations as follows:

$$\begin{aligned} &\bar{P}_1 = P_1(t_0), \quad \bar{P}_2(s) = P_2(t_0, s), \\ &\bar{W}_1 = W_1(t_0), \quad \bar{W}_2(s) = W_2(t_0, s). \end{aligned}$$

Finally, the receding horizon predictive control is represented as a distributed state feedback strategy as follows:

$$u_R(x_t) = \begin{cases} -B^T\left[\bar{P}_1 x(t) + \int_{-h}^{0} \bar{P}_2(s)x(t+s)ds\right], \ T_p > h \\ -B^T\left[\bar{W}_1 x(t) + \int_{-h}^{T_p-h} \bar{W}_2(s)x(t+s)ds\right], \ 0 < T_p \le h \end{cases}$$

It is noted that the feedback strategy needs only the state trajectories for time interval $[t-h, t]$ and is invariant with time.

Remark 2: In order to solve partial differential equations given above, we utilize a numerical procedure introduced in [10]. The main idea of the method in [10] is that the original partial differential equations can be transformed into ordinary differential equations by appropriate change of variables.

Now we have constructed a receding horizon predictive control from the solution to a finite horizon dynamic game problem. However, the only thing we can say about the control is that it is obtained based on the receding horizon strategy. Nothing can be said about the asymptotic stability and H_∞-norm boundedness. We therefore will investigate those issues in the next section.

3. Nonincreasing Monotonicity of a Saddle Point Value

Theorem 1: Given $\gamma > 0$, assume that there exist $X > 0$, S, Y, and Y_1 such that

$$\begin{bmatrix} (1,1) & A_1S+BY_1 & B_w & XQ^{\frac{1}{2}} & Y^T & X \\ \star & -S & 0 & 0 & Y_1^T & 0 \\ \star & \star & -\gamma^2 I & 0 & 0 & 0 \\ \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & -S \end{bmatrix} \leq 0. \tag{16}$$

where

$$(1,1) = (AX+BY)+(AX+BY)^T.$$

If we choose terminal weighting matrices F_1 and F_2 such that $F_1 = X^{-1}$ and $F_2 = S^{-1}$, the saddle point value $J^*(x_{t_0}, t_0, \sigma)$ satisfies the following nonincreasing monotonicity property:

$$\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \leq 0, \ \forall \sigma > t_0. \tag{17}$$

Proof:

$$\begin{aligned} & \frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \\ = \ & \frac{1}{\Delta}\Big\{ \int_{t_0}^{\sigma+\Delta} [\bar{x}^T(\tau)Q\bar{x}(\tau) + \bar{\mu}^T(\tau, x_\tau)\bar{\mu}(\tau, x_\tau) \\ & -\gamma^2 \bar{\nu}^T(\tau, x_\tau)\bar{\nu}(\tau, x_\tau)]d\tau \\ & +\bar{x}^T(\sigma+\Delta)F_1\bar{x}(\sigma+\Delta) + \int_{\sigma+\Delta-h}^{\sigma+\Delta} \bar{x}^T(\tau)F_2\bar{x}(\tau)d\tau \\ & -\int_{t_0}^{\sigma} [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{\mu}^T(\tau, x_\tau)\hat{\mu}(\tau, x_\tau) \\ & -\gamma^2 \hat{\nu}^T(\tau, x_\tau)\hat{\nu}(\tau, x_\tau)]d\tau + \hat{x}^T(\sigma)F_1\hat{x}(\sigma) \\ & -\int_{\sigma-h}^{\sigma} \hat{x}^T(\tau)F_2\hat{x}(\tau)d\tau \Big\} \end{aligned}$$

where the pair $(\bar{\mu}, \bar{\nu})$ is a saddle point solution for $J(x_{t_0}, t_0, \sigma+\Delta, u, w)$ and the pair $(\hat{\mu}, \hat{\nu})$ is for $J(x_{t_0}, t_0, \sigma, u, v)$. $\bar{x}$ denotes the state trajectory resulting from the strategies $\bar{\mu}$ and $\bar{\nu}$ and $\hat{x}$ denotes the state trajectory resulting from the strategies $\hat{\mu}$ and $\hat{\nu}$. Replace the strategy $\bar{\mu}$ and $\hat{\nu}$ by $\hat{\mu}$ and $\bar{\nu}$ up to σ and use $u(\tau) = Kx(\tau) + K_1x(\tau - h)$ and $w(\tau) = \hat{\nu}(\tau, x_\tau)$ for $\tau \geq \sigma$. It is noted that, since we have changed strategies, the resulting state trajectory is neither $\bar{x}$ nor $\hat{x}$. Let's denote the resulting state trajectory by x. Then we have

$$\begin{aligned} & \frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \\ \leq \ & \lim_{\Delta \to 0} \frac{1}{\Delta}\Big\{ \int_{\sigma}^{\sigma+\Delta} [x^T(\tau)Qx(\tau) + [Kx(\tau)+K_1x(\tau-h)]^T \\ & \times[Kx(\tau)+K_1x(\tau-h)] - \gamma^2 w^T(\tau)w(\tau)]d\tau \\ & +x^T(\sigma+\Delta)F_1x(\sigma+\Delta) + \int_{\sigma+\Delta-h}^{\sigma+\Delta} x^T(\tau)F_2x(\tau)d\tau \\ & -x^T(\sigma)F_1x(\sigma) - \int_{\sigma-h}^{\sigma} x^T(\tau)F_2x(\tau)d\tau \Big\} \\ = \ & x^T(\sigma)Qx(\sigma) + [Kx(\sigma)+K_1x(\sigma-h)]^T \\ & \times[Kx(\sigma)+K_1x(\sigma-h)] \\ & -\gamma^2 w^T(\tau)w(\tau) \\ & +\frac{d}{d\sigma}\Big\{ x^T(\sigma)F_1x(\sigma) + \int_{\sigma-h}^{\sigma} x^T(\tau)F_2x(\tau)d\tau \Big\} \\ = \ & x^T(\sigma)Qx(\sigma) + [Kx(\sigma)+K_1x(\sigma-h)]^T \\ & \times[Kx(\sigma)+K_1x(\sigma-h)] \\ & -\gamma^2 w^T(\tau)w(\tau) + 2\dot{x}^T(\sigma)F_1x(\sigma) \\ & +x^T(\sigma)F_2x(\sigma) - x^T(\sigma-h)F_2x(\sigma-h). \end{aligned}$$

After substituting $\dot{x}(\sigma) = (A+BK)x(\sigma) + (A_1+BK_1)x(\sigma-h) + B_ww(\sigma)$ into the above, we obtain

$$\begin{aligned} \frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \leq & \begin{bmatrix} x(\sigma) \\ x(\sigma-h) \\ w(\sigma) \end{bmatrix}^T \\ & \times \underbrace{\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & F_1B_w \\ \star & -F_2+K_1^TK_1 & 0 \\ \star & \star & -\gamma^2 I \end{bmatrix}}_{\Lambda} \begin{bmatrix} x(\sigma) \\ x(\sigma-h) \\ w(\sigma) \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \Lambda_{11} &= (A+BK)^TF_1 + F_1(A+BK) + Q + K^TK + F_2 \\ \Lambda_{12} &= F_1(A_1+BK_1) + K^TK_1. \end{aligned}$$

It is apparent that, if $\Lambda \leq 0$, nonincreasing monotonicity in (17) holds. $\Lambda \leq 0$ can be rewritten as follows:

$$\begin{aligned} & \begin{bmatrix} (A+BK)^TF_1 + F_1(A+BK) & F_1(A+BK_1) \\ \star & -F_2 \end{bmatrix} \\ + & \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ K & K_1 \\ I & 0 \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & F_2^{-1} \end{bmatrix}^{-1} \\ \times & \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ K & K_1 \\ I & 0 \end{bmatrix} \leq 0. \end{aligned}$$

Pre- and post-multiply the above matrix inequality by $\mathrm{diag}\{F_1^{-1}, F_2^{-1}\}$. The above inequality is then equivalently changed into (16) according to the Schur complement. This completes the proof. ■

Remark 3: The nonincreasing monotonicity of the saddle point value implies that the saddle point value does not increase even though we increase the horizon length. As will be shown in the next section, this property plays an important role in RHC's achieving closed-loop stability and H_∞-norm boundedness.

4. Asymptotic Stability and H_∞-norm Boundedness

In this section, we show that the proposed receding horizon predictive control achieves the closed-loop asymptotic stability for zero disturbance and satisfies the H_∞-norm boundedness for zero initial condition.

Theorem 2: Given $Q > 0$ and $\gamma > 0$, if $\frac{J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \leq 0$ for $\sigma > t_0$, the system (1) controlled by the RHHPC is asymptotically stable for zero disturbance and satisfies infinite horizon H_∞-norm bound for zero initial condition.

Proof:

$$
\begin{aligned}
& J^*(x_t,t,t+T_p) \\
= & \int_t^{t+\theta}[x^T(\tau)Qx(\tau)+\mu^{*T}(\tau,x_\tau)\mu^*(\tau,x_\tau) \\
& -\gamma^2\nu^{*T}(\tau,x_\tau)\nu^*(\tau,x_\tau)]d\tau+J^*(x_{t+\theta},t+\theta,t+T_p)
\end{aligned}
$$

where the pair (μ^*,ν^*) is a saddle point solution for $J(x_t,t,t+T_p,u,w)$. Replace the saddle point strategy $\nu^*(\tau,x_\tau)$ by $\nu(\tau,x_\tau)$ given by

$$
\nu(\tau,x_\tau)=\begin{cases} w(\tau) & , \quad t\le\tau<t+\theta \\ \nu^*(\tau,x_\tau) & , \quad t+\theta\le\tau\le t+T_p \end{cases}
$$

where $w(\tau)$ denotes an arbitrary signal. Recalling $J^*(x_t,t,t+T_p)=J(x_t,t,t+T_p,\mu^*,\nu^*)$, we obtain

$$
\begin{aligned}
& J^*(x_t,t,t+T_p) \\
\ge & J(x_t,t,t+T_p,\mu^*,\nu) \\
= & \int_t^{t+\theta}[\bar{x}^T(\tau)Q\bar{x}(\tau)+\mu^T(\tau,\bar{x}_\tau)\mu(\tau,\bar{x}_\tau)-\gamma^2w^T(\tau)w(\tau)]d\tau \\
& +J(\bar{x}_{t+\theta},t+\theta,t+T_p,\mu^*,\nu^*)
\end{aligned}
$$

where $\bar{x}$ denotes the state trajectory resulting from the strategy μ^* and ν. It is noted that the strategy pair (μ^*,ν^*) is the saddle point solution not only for $J(x_t,t,t+T_p,u,w)$ but also for $J(x_{t+\theta},t+\theta,t+T_p,u,w)$ because the terminal time is $t+T_p$ for both cases. Therefore $J(\bar{x}_{t+\theta},t+\theta,t+T_p,\mu^*,\nu^*)=J^*(\bar{x}_{t+\theta},t+\theta,t+T_p)$. Furthermore, from the fact that $\frac{J^*(x_{t_0},t_0,\sigma)}{\partial\sigma}\le 0$ for $\sigma>t_0$, it follows $J^*(\bar{x}_{t+\theta},t+\theta,t+T_p)\ge J^*(\bar{x}_{t+\theta},t+\theta,t+T_p+\theta)$. This, in turn, leads to

$$
\begin{aligned}
& J^*(x_t,t,t+T_p)\ge \\
& \int_t^{t+\theta}[\bar{x}^T(\tau)Q\bar{x}(\tau)+\mu^{*T}(\tau,\bar{x}_\tau)\mu^*(\tau,\bar{x}_\tau)-\gamma^2w^T(\tau)w(\tau)]d\tau \\
& +J^*(\bar{x}_{t+\theta},t+\theta,t+T_p+\theta).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
& \frac{J^*(\bar{x}_{t+\theta},t+\theta,t+T_p+\theta)-J^*(x_t,t,t+T_p)}{\theta} \\
& \quad \le -\frac{1}{\theta}\int_t^{t+\theta}[\bar{x}^T(\tau)Q\bar{x}(\tau)+\mu^{*T}(\tau,\bar{x}_\tau)\mu^*(\tau,\bar{x}_\tau) \\
& \qquad -\gamma^2w^T(\tau)w(\tau)]d\tau
\end{aligned}
$$

When $\theta\to 0$, we have

$$
\begin{aligned}
& \frac{dJ^*(x_t,t,t+T_p)}{dt} \\
\le & -[x^T(t)Qx(t)+u_R^T(x_t)u_R(x_t)-\gamma^2w^T(t)w(t)] \quad (18) \\
= & -\psi^T\begin{bmatrix} Q+\bar{P}_1BB^T\bar{P}_1 & \bar{P}_1B \\ B^T\bar{P}_1 & I \end{bmatrix}\psi-\gamma^2w^T(t)w(t)
\end{aligned}
$$

where

$$
\psi=\begin{bmatrix} x(t) \\ B^T\int_{-h}^0\bar{P}_2(s)x(t+s)ds \end{bmatrix}.
$$

In case of $w(t)=0$, we know that

$$
\begin{aligned}
\frac{dJ^*(t,t,t+T_p)}{dt} & \le -\psi^T\begin{bmatrix} Q+\bar{P}_1BB^T\bar{P}_1 & \bar{P}_1B \\ B^T\bar{P}_1 & I \end{bmatrix}\psi \\
& \le -x^T(t)Qx(t) \\
& \le -\lambda_{min}(Q)\|x(t)\|^2
\end{aligned}
$$

We can conclude that $J^*(x_t,t,t+T_p)$ is a Lyapunov-Krasovskii functional from the above derivation. Therefore, the closed-loop system is asymptotically stable. We then prove that H_∞-norm bound is guaranteed for the closed-loop system. Consider an infinite horizon cost J_w given by

$$
J_w=\int_0^\infty[x^T(t)Qx(t)+u_R^T(x_t)u_R(x_t)-\gamma^2w^T(t)w(t)]dt.
$$

We have only to show that $J_w\le 0$ for the proof of H_∞-norm bound.

$$
\begin{aligned}
J_w = & \int_0^\infty[x^T(t)Qx(t)+u_R^T(x_t)u_R(x_t)-\gamma^2w^T(t)w(t) \\
& +\frac{d}{dt}J^*(x_t,t,t+T_p)]dt \\
& +J^*(x_0,0,T_p)-J^*(x_t,t,t+T_p)|_{t=\infty}
\end{aligned}
$$

For zero initial condition, i.e. $x(s)=0, s\in[-h,0]$, $J^*(x_0,0,T_p)$ is equal to zero. Furthermore, since the saddle point value is nonnegative-definite, we have $J^*(x_t,t,t+T_p)|_{t=\infty}\ge 0$. This, in turn, leads to

$$
\begin{aligned}
J_w \le & \int_0^\infty[x^T(t)Qx(t)+u_R^T(x_t)u_R(x_t)-\gamma^2w^T(t)w(t) \\
& +\frac{d}{dt}J^*(x_t,t,t+T_p)]dt
\end{aligned}
$$

The integrand above is less than or equal to zero from (18). Consequently we can conclude that $J_w\le 0$. This completes the proof. ■

Theorem 2 states that the nonincreasing monotonicity of the saddle point value is the sufficient condition for the closed-loop stability and the H_∞-norm boundedness. An LMI condition on the terminal weighting matrices under which the saddle point satisfies nonincreasing monotonicity was given in Theorem 1. Therefore, we can construct the following corollary:

Corollary 2: Given $Q>0$ and $\gamma>0$, if the LMI (16) is feasible and we can obtain two terminal weighting matrices F_1 and F_2, the system (1) controlled by the proposed RHHPC is asymptotically stable for zero disturbance and satisfies infinite horizon H_∞ norm bound for zero initial condition.

5. Numerical Example

In this section, we provide a numerical example in order to illustrate the properties of the proposed RHHPC. Consider a chemical reactor system taken from [11]. The system matrices are given by

$$
A=\begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ -6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix},
$$

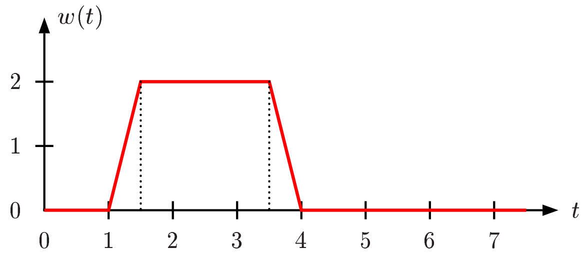

Fig. 1. Disturbance, $w(t)$

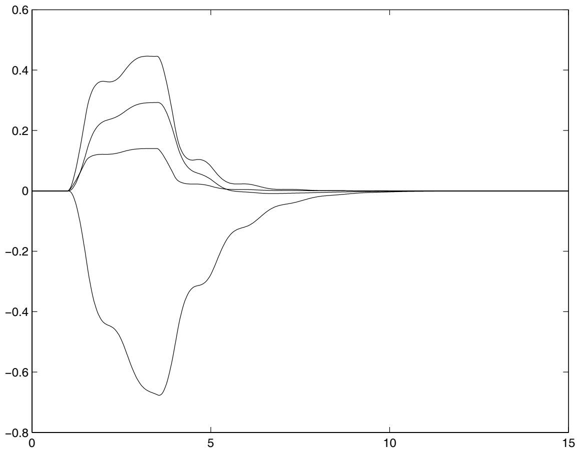

Fig. 2. State response to the disturbance input

$$A_1 = \text{diag}\{1.92, 1.92, 1.87, 0.724\}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We chose $Q = I$. For $\gamma = 0.85$, we obtained terminal weighting matrices satisfying the monotonicity condition in (16) as follows:

$$F_1 = \begin{bmatrix} 1.8193 & -0.6531 & 0.2726 & -0.0417 \\ -0.6531 & 0.3979 & -0.1397 & 0.0605 \\ 0.2726 & -0.1397 & 0.1042 & 0.0467 \\ -0.0417 & 0.0605 & 0.0467 & 0.2169 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 3.2895 & -1.3429 & 0.5725 & 0.0081 \\ -1.3429 & 0.6921 & -0.2584 & 0.0258 \\ 0.5725 & -0.2584 & 0.1380 & 0.0301 \\ 0.0081 & 0.0258 & 0.0301 & 0.0466 \end{bmatrix}$$

In order to illustrate the stability and the H_∞-norm boundedness, we applied the disturbance input $w(t)$ that has the shape in Figure 1. Figure 2 shows the state response of the closed-loop system to the disturbance in Figure 1. It clearly shows that the resulting closed-loop system is stable. The value of $\|z\|_2/\|w\|_2$ was computed to be 0.5384, which is less than $\gamma = 0.85$. This supports that the closed-loop system satisfies H_∞-norm boundedness.

6. Conclusions

In this paper, we proposed the receding horizon H_∞ predictive control (RHHPC) for linear systems with a state-delay. Firstly, we proposed a new cost function for a dynamic game problem. The terminal weighting term is parameterized by two terminal weighting matrices. Secondly, we derived a saddle point solution to a finite horizon dynamic game problem using the proposed cost function. Thirdly, a receding horizon predictive control was constructed from the obtained solution. We showed that, under the nonincreasing monotonicity condition of a saddle point value, the proposed receding horizon predictive control is stabilizing and satisfies the H_∞-norm bound. We proposed an LMI condition on the terminal weighting matrices, under which the saddle point value satisfies the nonincreasinn monotonicity. Main work of this paper is the development of the RHHPC for time-delay systems.

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