Chaos Synchronization Using Error Feedback Coupling

Behzad Khademian, and Mohammad Haeri

Advanced Control System Lab., Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran
(Tel : +98-21-616-5971; E-mail: khademian@mehr.sharif.edu)
(Tel : +98-21-616-5964; E-mail: haeri@sina.sharif.edu)

Abstract: This paper presents synchronization of two identical Modified Chua's circuits using two strategies of error feedback coupling. In the first method the synchronization is achieved by linear unidirectional and in the second one by linear bidirectional error feedback coupling. Both proposed methods can make the states of the Modified Chua's circuits globally asymptotically synchronized. Numerical results are provided to show the effectiveness of the proposed approaches and to compare them together based on different criteria.

Keywords: Chaos, synchronization, modified Chua' circuits, feedback coupling, unidirectional, bidirectional

1. INTRODUCTION

Synchronization phenomena in coupled chaotic systems have been extensively studied during the last decade due to its theoretical challenge and its great potential applications in secure communications, chemical reactions and modeling brain activity [1-3]. However, some many synchronization methods require several controllers to realize synchronization. The linearly error feedback coupling scheme can be adopted in many real fields, due to its simple configuration and easy implementation. According to the condition of coupling signals, they can be classified into bidirectional [4-6] and unidirectional [7-9].

The objective of the synchronization is that the error states of two chaotic systems are asymptotically stable in the Lyapunov sense. In this paper by using Lyapunov stability theorem and linear matrix inequality, we stabilize the error between chaotic drive and response systems at the origin, via unidirectional and bidirectional linear feedback approaches.

The paper is organized as follows. In Section 2, based on Lyapunov stability theorem and linear matrix inequality, the coupling parameters for linearly bidirectional coupled chaotic systems are derived. In Section 3, these methods are applied to a coupled modified Chua's circuits. Numerical simulations are carried out in Section 4 for illustration and verification of the methods. Finally some concluding remarks are given in Section 5.

2. CHAOS SYNCHRONIZATION WITH BIDIRECTIONAL OR UNIDIRECTIONAL COUPLING

In order to observe the synchronization behavior, we consider two identical n-dimensional chaotic systems (which are called derive and response systems). The initial conditions of the systems are different. The drive and response systems are described by Equations (1) and (2) respectively. The third term of the right hand side has been added to the systems dynamics for the control design purposes.

$$\dot{x} = Ax + f(x) + K_1(y - x) \tag{1}$$

and

$$\dot{y} = Ay + f(y) + K_2(x - y) \tag{2}$$

$A \in R^{n\times n}$ is a constant matrix, $f : R^n \to R^n$ is a continuous nonlinear function, K_1 and K_2 are diagonal matrices which are used as the feedback gains to be calculated.

Considering $e = y - x$ and $f(y) - f(x) = M_{x,y}e$, error dynamics can be written as:

$$\dot{e} = (A + M_{x,y} - (K_1 + K_2))e \tag{4}$$

Since a chaotic system has bounded trajectories, $M_{x,y} = M(x, y)$ is a bounded matrix.

The following theorem [10] gives the sufficient condition for system (4) to be globally stable.

Theorem 1. If there exists a positive definite symmetric constant matrix P and a constant $\varepsilon > 0$, such that

$$(A+M_{x,y} - (K_1 + K_2))^T P + P(A+M_{x,y} - (K_1 + K_2)) < -\varepsilon I \tag{5}$$

uniformly for any x and y, where I is the identity matrix, the error dynamics system (4) is globally stable, i.e., system (1) and (2) are synchronized.

Proof. Consider the following Lyapunov function

$$V(t) = e^T(t)Pe(t)$$

Calculating its derivative, we have

$$\begin{aligned}\dot{V} &= \dot{e}^T Pe + e^T P\dot{e} = e^T (A + M_{x,y} - (K_1 + K_2))^T Pe + \\ &\quad e^T P(A + M_{x,y} - (K_1 + K_2))^T e \\ &= e^T((A + M_{x,y} - (K_1 + K_2))^T P + \\ &\quad P(A + M_{x,y} - (K_1 + K_2)))e \le -\varepsilon e^T e < 0\end{aligned}$$

for all $e(t) \neq 0$. So, the theorem is proved. □

With respect to Theorem 1, the result can be simplified as the condition of unidirectional coupling chaos systems when $K_1 = 0$ or $K_2 = 0$.

In theorem 2 [11] the sufficient condition for system (4) to be globally asymptotically synchronized, is discussed.

Theorem 2. If there exists the feedback gain matrices K_1 and K_2 such that

$$\int_{t_0}^{+\infty} \mu(A + M_{x,y} - (K_1 + K_2))dt = -\infty \qquad \forall t_0 \ge 0$$

uniformly for all x, y, then the error dynamical system (4) is globally asymptotically stable at the origin, implying that the two systems (1) and (2) are globally asymptotically synchronized.

$\mu(*)$ denotes the matrix measure derive from the matrix norm $\|*\|$.

Corollary 1. The two systems (1) and (2) are globally asymptotically synchronized, if there exists the feedback gain matrix $K = K_1 + K_2$ such that at lease one of the following conditions is satisfied.

i) $\int_{t_0}^{+\infty} \max_j (a_{jj} + M_{jj}(x,y) - K_j + \sum_{i=1, i\neq j}^{n} |a_{ij} + M_{ij}(x,y)|) dt = -\infty$

ii) $\int_{t_0}^{+\infty} \max_i (a_{ii} + M_{ii}(x,y) - K_i + \sum_{j=1, j\neq i}^{n} |a_{ji} + M_{ji}(x,y)|) dt = -\infty$

3. SYNCHRONIZATION OF MODIFIED CHUA'S CIRCUIT

In this section we apply the above techniques to modified Chua's circuit described by [12];

$$\begin{aligned} \dot{x} &= p(y - f(x)), \quad f(x) = 2x^3 - x/7 \\ \dot{y} &= x - y + z \\ \dot{z} &= -qy \end{aligned} \tag{6}$$

which has a chaotic attractor as shown in Fig. 1 when $p = 10$ and $q = 100/7$.

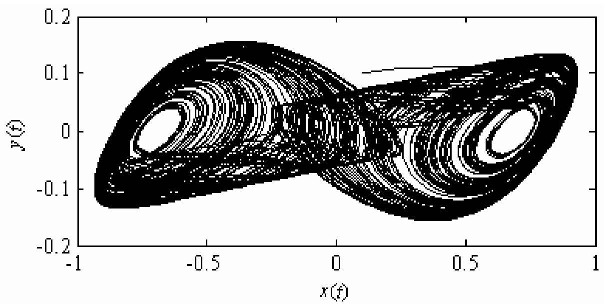

Fig. 1. The modified Chua's circuit chaotic attractor.

We have two modified Chua's circuits where the drive system with three state variables denoted by the subscript 1 drives the response system having identical equations denoted by the subscript 2. However, the initial condition on the drive system is different from that of the response system. The two modified Chua's circuits with linearly bidirectional coupling are described, respectively, by the following equations.

$$\begin{aligned} \dot{x}_1 &= p(y_1 - (2x_1^3 - x_1)/7) + k_{11}(x_2 - x_1) \\ \dot{y}_1 &= x_1 - y_1 + z_1 + k_{12}(y_2 - y_1) \\ \dot{z}_1 &= -qy_1 + k_{13}(z_2 - z_1) \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{x}_2 &= p(y_2 - (2x_2^3 - x_2)/7) + k_{21}(x_1 - x_2) \\ \dot{y}_2 &= x_2 - y_2 + z_2 + k_{22}(y_1 - y_2) \\ \dot{z}_2 &= -qy_2 + k_{23}(z_1 - z_2) \end{aligned} \tag{8}$$

Where k_{ij} $(i = 1,2;\ j = 1,2,3)$ are coupling parameters that to be determined in order to synchronize two modified Chua's circuits in spite of the differences in the initial conditions.

Let $e_x = x_2 - x_1, e_y = y_2 - y_1, e_z = z_2 - z_1$. Then, error dynamics can be written as;

$$\begin{aligned} \dot{e}_x &= p(e_y - 2e_x(M_{x_1,x_2} + 1)/7) - k_1 e_x \\ \dot{e}_y &= e_x - e_y + e_z - k_2 e_y \\ \dot{e}_z &= -qe_y - k_3 e_z \end{aligned} \tag{9}$$

where $k_i = k_{1i} + k_{2i}, i = 1, 2, 3$ and $M_{x_1,x_2} = x_1^2 + x_2^2 + x_1 x_2$ is a bounded function. From Fig. 1 it can be seen that $-1 < x(t) < 1$.

Let choose the positive definite symmetric constant matrix $P = diag\{p_1, p_2, p_3\}$ $p_i > 0,\ i = 1, 2, 3$ and any constant $\varepsilon > 0$, then according to theorem 1;

$$(A + M_{x_1,x_2} - (K_1 + K_2))^T P + P(A + M_{x_1,x_2} - (K_1 + K_2)) + \varepsilon I$$
$$= \begin{pmatrix} -2p_1(k_1 - \frac{p}{7} + 2pM_{x_1,x_2} - \frac{\varepsilon}{2p_1}) & pp_1 + p_2 & 0 \\ pp_1 + p_2 & -2p_2(1 + k_2 - \frac{\varepsilon}{2p_2}) & p_2 - qp_3 \\ 0 & p_2 - qp_3 & -2p(k_3 - \frac{\varepsilon}{2p_3}) \end{pmatrix} \tag{10}$$

Matrix in (10) is negative definite if and only if the following inequalities satisfy;

$$\Delta_1 = -2p_1(k_1 - \frac{p}{7} + 2pM_{x_1,x_2} - \frac{\varepsilon}{2p_1}) > 0 \tag{11}$$

$$\Delta_2 = \begin{vmatrix} \Delta_1 & pp_1 + p_2 \\ pp_1 + p_2 & -2p_2(1 + k_2 - \frac{\varepsilon}{2p_2}) \end{vmatrix} < 0 \tag{12}$$

$$\Delta_3 = \begin{vmatrix} \Delta_1 & pp_1 + p_2 & 0 \\ pp_1 + p_2 & -2p_2(1 + k_2 - \frac{\varepsilon}{2p_2}) & p_2 - qp_3 \\ 0 & p_2 - qp_3 & -2p(k_3 - \frac{\varepsilon}{2p_3}) \end{vmatrix} > 0 \tag{13}$$

By calculating the above determinants we can extract the following inequalities to get the values of $k_i = k_{1i} + k_{2i}$, $i = 1, 2, 3$;

$$k_1 > \frac{p}{7} - 2pM_{x_1,x_2} + \frac{\varepsilon}{2p_1} \tag{14}$$

$$k_2 < \frac{(pp_1 + p_2)^2}{4p_1 p_2 (k_1 - \frac{p}{7} + 2pM_{x_1,x_2} - \frac{\varepsilon}{2p_1})} + \frac{\varepsilon}{2p_2} - 1 \tag{15}$$

$$k_3 > \frac{\Delta_1(p_2 - qp_3)}{2p((pp_1 + p_2)^2 - 2p_2\Delta_1(1 + k_2 - \frac{\varepsilon}{2p_2}))} + \frac{\varepsilon}{2p_3} \tag{16}$$

According to Theorem 2 and Corollary 1 there is a more simple procedure for choosing the feedback gain matrices K_1 and K_2 as follows:

$$A + M_{x_1,x_2} - (K_1 + K_2)$$

$$= \begin{pmatrix} -2p(M_{x_1,x_2} + 1)/7 - k_1 & p & 0 \\ 1 & -1 - k_2 & 1 \\ 0 & -q & -k_3 \end{pmatrix} \quad (17)$$

Now we can choose $k_i = k_{1i} + k_{2i}$, $i = 1, 2, 3$ such that

$$\max\{-2p(M_{x_1,x_2} + 1)/7 - k_1 + p, -k_2 + 1, q - k_3\} < 0 \quad (18)$$

or

$$\max\{-2p(M_{x_1,x_2} + 1)/7 - k_1 + 1, p - 1 - k_2 + q, 1 - k_3\} < 0 \quad (19)$$

So the two coupled modified Chua's circuits are globally asymptotically synchronized.

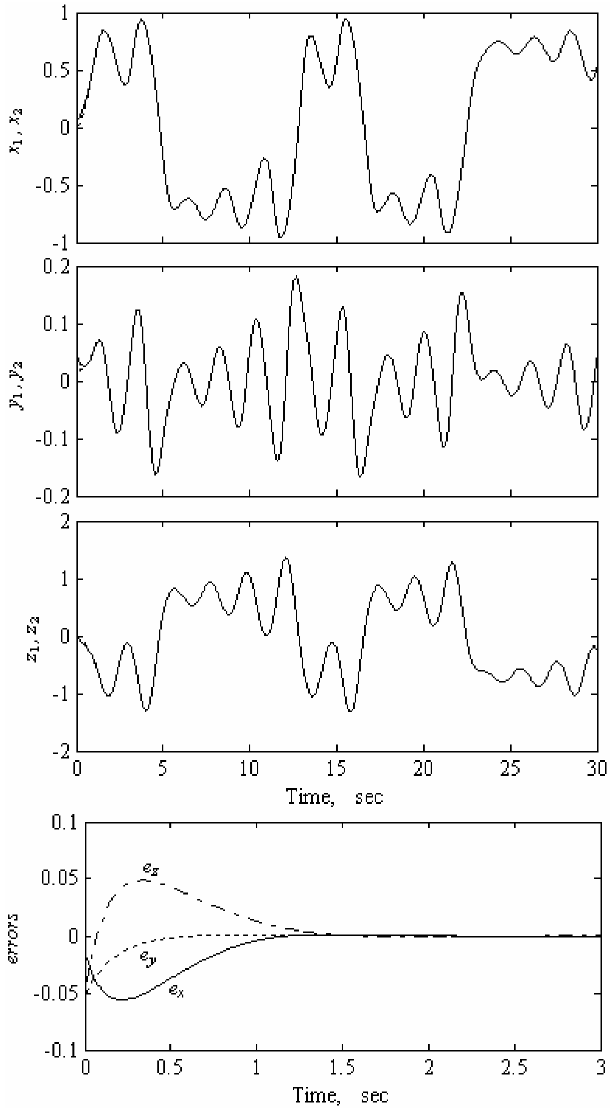

Fig. 2. Synchronization of linearly bidirectional coupled modified Chua's circuits.

4. NUMERICAL RESULTS ES

To verify the effectiveness of the proposed synchronization approaches, we did some numerical simulations. The initial values of drive system and response system in all simulations are taken $(x_1(0) = 0.02, y_1(0) = 0.05, \quad z_1(0) = 0.04)$ and $(x_2(0) = 0.0002 \quad y_2(0) = 0.0005, \quad z_2(0) = 0.0004)$ respectively. Due to the Theorem 1 and by selecting $P = diag\{1, 1, 1\}$, $\varepsilon = 0.01$ and the systems parameters $p = 10$ and $q = 100/7$, we can choose coupling parameters $k_{11} = k_{21} = 2$, $k_{12} = k_{22} = 3$, $k_{13} = k_{23} = 1$ which can satisfy the inequalities (14), (15) and (16). The synchronized states of the drive and response systems and error trajectories are shown in Fig.2.

Selecting the matrix $K_2 = 0$, the unidirectional coupled modified Chua's circuits can be synchronized by choosing $k_{11} = 2$, $k_{12} = 3$, $k_{13} = 1$. As it is shown in Fig. 3, the states of two uncertain modified Chua's circuits are synchronized. Due to the inequalities (18) or (19) for unidirectional method, we can choose $K_2 = 0$, $k_{11} = 1$, $k_{12} = 30$, $k_{13} = 30$. The simulation results are shown in Fig.4.

Comparing Figs. 3 and 4, we can see that with larger values of coupling parameters, the states are being synchronized more quickly.

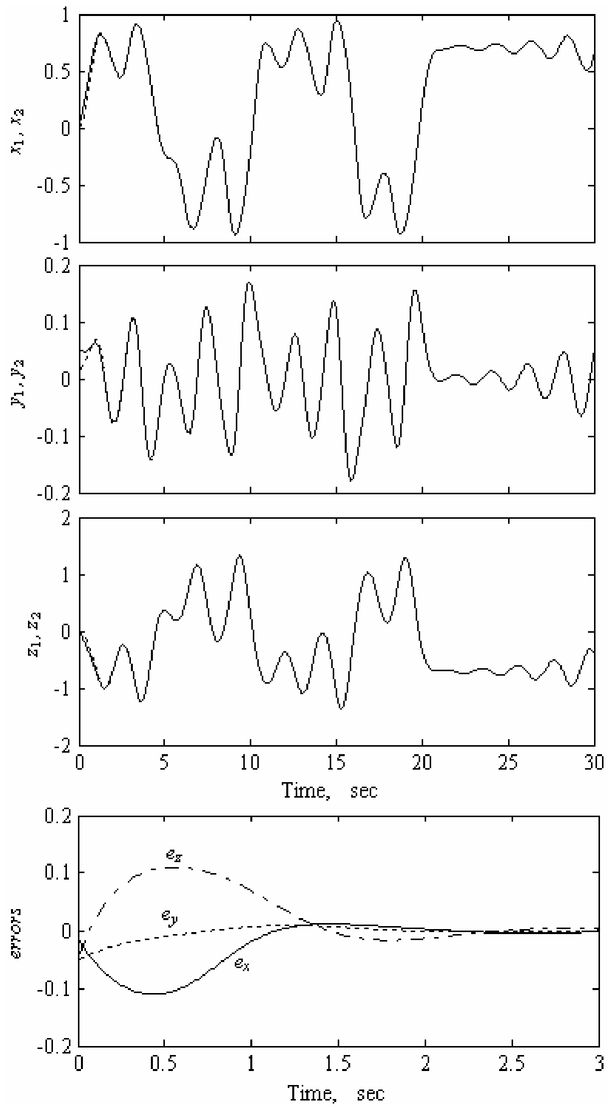

Fig. 3. Synchronization of linearly unidirectional coupled modified Chua's circuits.

To verify the robustness of the proposed methods in presence of noise, white Gaussian noise with mean 0 and variance 0.01 is added to the drive system states. The coupling parameters are taken $k_{11} = k_{21} = 2$, $k_{12} = k_{22} = 3$, $k_{13} = k_{23} = 1$ in bidirectional coupling and $K_2 = 0$, $k_{11} = 2$, $k_{12} = 3$, $k_{13} = 1$ in unidirectional coupling. As it is shown in Figs. 5 and 6, both methods are stable and the error values are bounded.

5. CONCLUSION

In this paper the unidirectional and bidirectional synchronization methods have been developed and applied to a coupled modified Chua's circuits. Based on Lyapunov stability theorem and linear matrix inequality the values of coupling parameters are derived. Finally the numerical results are presented to verify and to compare the proposed methods.

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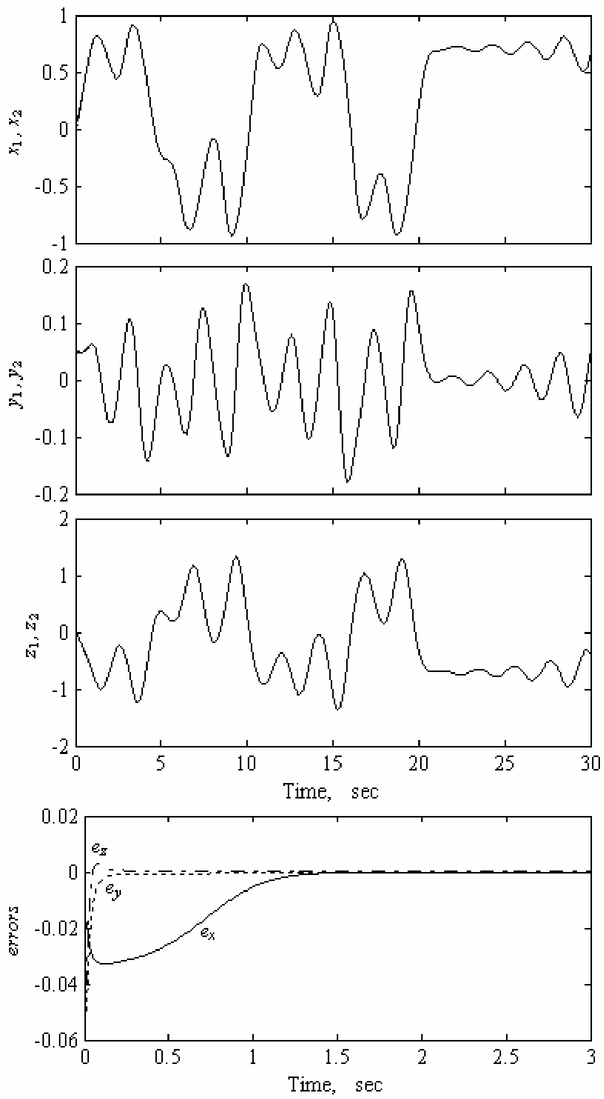

Fig. 4. Synchronization of linearly unidirectional coupled modified Chua's circuits with large value of coupling parameters.

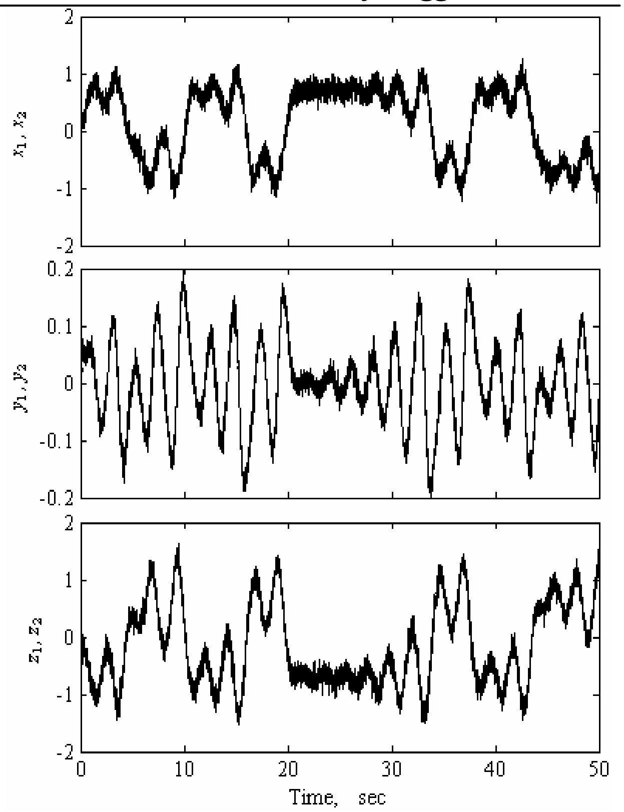

Fig. 5. Synchronization of linearly unidirectional coupled modified Chua's circuits in presence of noise.

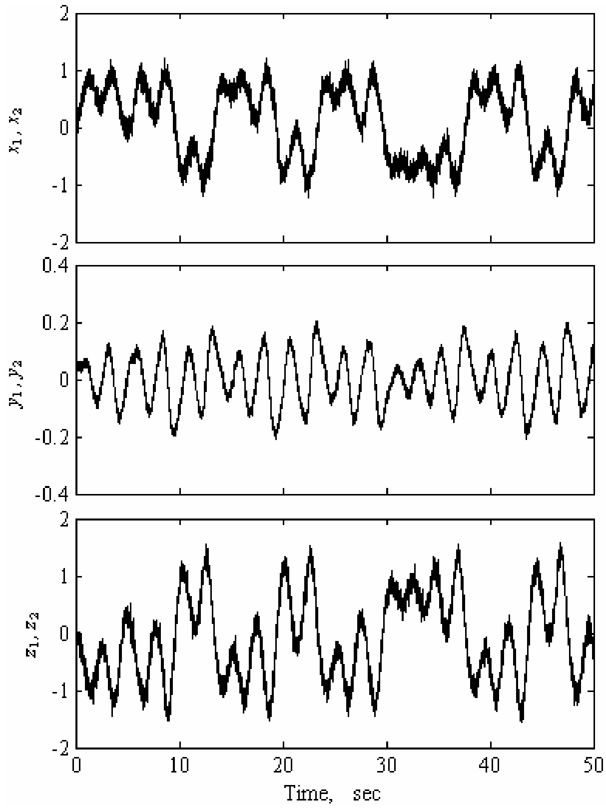

Fig. 6. Synchronization of linearly bidirectional coupled modified Chua's circuits in presence of noise.

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