

# Least-squares Lattice Laguerre Smoother

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**Abstract:** This paper introduces the least-squares order-recursive lattice (LSORL) Laguerre smoother that has order-recursive smoothing structure based on the Laguerre signal representation. The LSORL Laguerre smoother gives excellent performance for a channel equalization problem with smaller order of tap-weights than its counterpart algorithm based on the transversal filter structure. Simulation results show that the LSORL Laguerre smoother gives better performance than the LSORL transversal smoother.

**Keywords:** Laguerre filter, smoother, least-squares estimation, adaptive filters

## 1 Introduction

Scattering channels, including wireless indoor channels, acoustic room channels, and damped-long impulse response systems, have significant multipath problem which leads to lots of efforts for characterizing such channels [1]. Almost measurements of scattering channels show that the channels have long and oscillatory channel impulse responses. In case of wireless data communications, a receiver should employ a channel equalization to eliminate interferences between adjacent symbols transmitted through the channels from a transmitter. Especially, it is well-known that non-causal channel equalization by delaying signals is required to consider the pre-cursor interference as well as the post-cursor, which requires smoothing, rather than filtering, for the channel equalization. The LSORL transversal smoother, which has some advantages such as computational efficiency, stage-to-stage modularity, and numerical robustness, was introduced in [2]. However, the transversal structure is inadequate for the channel equalization because it requires plenty of filter-tap orders to equalize such channels. The Laguerre sequences are excellent for characterizing long and lowpass-filtered oscillatory impulse responses [3], thus this characteristics is suitable for the scattering channel equalization. Recently, [3] proposed an RLS Laguerre prediction algorithm having the lattice structure. By extending the work, this paper introduces the LSORL Laguerre smoother that is suitable for equalizing above-mentioned channels.

## 2 Main results

Laguerre sequence is known as  $L_i(z, a) \triangleq \sqrt{1-a^2}(z^{-1} - a)^i(1 - az^{-1})^{-(i+1)} = L_0(z, a)A^i(z, a)$ , where  $A(z, a) \triangleq (z^{-1} - a)(1 - az^{-1})^{-1}$ . As in Figure 1, the Laguerre filter output is written as  $y_M(n, a) = \sum_{i=0}^{M-1} w_i(n, a)u_i(n, a)$ . We shall obtain the least-squares order-recursive lattice Laguerre smoother by exploiting data structure that comes from the Laguerre filter proposed in [3]. For convenience, we shall henceforth fix the scale parameter  $a$  as a certain value; how to determine the best “ $a$ ” is beyond scope of this paper. The least-squares cost function for  $M(\triangleq p + f)$ -th order smooth-

ing error is defined as

$$\begin{aligned} J_M^f(n) &\triangleq \mathbf{e}_{p,f}^H(n)\mathbf{e}_{p,f}(n), \\ \mathbf{e}_{p,f}(n) &\triangleq \mathbf{d}_f(n) - \mathbf{H}_M(n)\mathbf{w}_{p,f}(n), \end{aligned}$$

where

$$\begin{aligned} \mathbf{d}_f(n) &\triangleq [d_f(0) \ \cdots \ d_f(n)]^T, \\ \mathbf{w}_{p,f}(n) &\triangleq [w_0(n) \ \cdots \ w_{M-1}(n)]^T, \end{aligned}$$

and  $H_M(n)$  is a data matrix defined in (3). Then, the two  $(M+1)$ -th order forward and backward smoothing errors [4] are

$$\mathbf{e}_{p,f+1}(n) = \mathbf{d}_{f+1}(n) - \mathbf{H}_{M+1}(n)\mathbf{w}_{p,f+1}(n), \quad (1)$$

$$\mathbf{e}_{p+1,f}(n) = \mathbf{d}_f(n) - \mathbf{H}_{M+1}(n)\mathbf{w}_{p+1,f}(n). \quad (2)$$

The difference of the two smoothing errors is that they use different desired vectors and thus different weight vectors. To obtain the order-update recursion for the forward smoothing error (1), we partition the  $(M+1)$ -th order data matrix as

$$\begin{aligned} H_{M+1}(n) &= \begin{bmatrix} u_0(0) & u_1(0) & \cdots & u_M(0) \\ \vdots & \vdots & \ddots & \vdots \\ u_0(n) & u_1(n) & \cdots & u_M(n) \end{bmatrix} \\ &= [\mathbf{u}_0(n) \ \bar{\mathbf{H}}_M(n)]. \end{aligned} \quad (3)$$

The optimal estimate of the desired vector,  $\hat{\mathbf{d}}_{f+1}(n)$ , can be written as

$$\begin{aligned} \hat{\mathbf{d}}_{p,f+1}(n) &= H_{M+1}(n)(H_{M+1}^H(n)H_{M+1}(n))^{-1} \cdot \\ &\quad H_{M+1}^H(n)\mathbf{d}_{f+1}(n). \end{aligned} \quad (4)$$

Using the above data matrix partition, the inverse of  $H_{M+1}^H(n)H_{M+1}(n) \triangleq R_{M+1}(n)$  is written as

$$R_{M+1}^{-1}(n) = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \bar{R}_M^{-1}(n) \end{bmatrix} + \frac{1}{\xi_M^f(n)} \begin{bmatrix} 1 \\ -\hat{\mathbf{w}}_M^f(n) \end{bmatrix} \begin{bmatrix} 1 & -\hat{\mathbf{w}}_M^{fH}(n) \end{bmatrix}, \quad (5)$$

where  $\hat{\mathbf{w}}_M^f(n)$  satisfies

$$\begin{aligned} \bar{R}_M(n)\hat{\mathbf{w}}_M^f(n) &= \bar{H}_M^H(n)\mathbf{u}_0(n), \\ \xi_M^f(n) &= \mathbf{f}_M^H(n)\mathbf{f}_M(n), \\ \mathbf{f}_M(n) &= \mathbf{u}_0(n) - \bar{\mathbf{H}}_M(n)\hat{\mathbf{w}}_M^f(n). \end{aligned}$$

If we substitute (5) for  $R_{M+1}^{-1}$  in (4), we get the following recursion

$$\begin{aligned}\hat{\mathbf{d}}_{p,f+1}(n) &= \bar{H}_M(n)\hat{\mathbf{w}}_{p,f+1}(\mathbf{n}) \\ &\quad + \mathbf{f}_M^H(\mathbf{n})\mathbf{d}_{f+1}(n)\xi_M^{-f}(\mathbf{n})\mathbf{f}_M(\mathbf{n}), \\ \hat{\mathbf{w}}_{p,f+1}(\mathbf{n}) &= \bar{R}_M^{-1}(n)\bar{H}_M^H(n)\mathbf{d}_{f+1}(\mathbf{n}).\end{aligned}$$

Thus, we get an order-update recursion for the forward smoothing error,

$$\mathbf{e}_{p,f+1}(\mathbf{n}) = \bar{\mathbf{e}}_{p,f+1}(\mathbf{n}) - \mathbf{f}_M^H(\mathbf{n})\mathbf{d}_{f+1}(n)\xi_M^{-f}(\mathbf{n})\mathbf{f}_M(\mathbf{n}), \quad (6)$$

where  $\bar{\mathbf{e}}_{p,f+1}(n) = \mathbf{d}_{f+1}(n) - \bar{H}_M(n)\hat{\mathbf{w}}_{p,f+1}(\mathbf{n})$ . Now, we shall explain the relation between  $\mathbf{e}_{p,f+1}(\mathbf{n})$  and  $\bar{\mathbf{e}}_{p,f}(n)$ . Using the following lower triangular Toeplitz matrix as suggested in [3],

$$\Phi(n) = \begin{bmatrix} -a & & & & \\ 1-a^2 & -a & & & \\ \vdots & \ddots & \ddots & & \\ a^{n-1}(1-a^2) & \cdots & 1-a^2 & -a & \end{bmatrix},$$

it holds that  $\bar{H}_M(n) = \Phi(n)H_M(n)$ , and  $\mathbf{d}_{f+1}(\mathbf{n}) = \Phi(\mathbf{n})\mathbf{d}_f(\mathbf{n})$ . With these relations, we can write

$$\begin{aligned}\hat{\mathbf{w}}_{p,f+1}(\mathbf{n}) &= \left(\mathbf{H}_M^H(\mathbf{n})\Phi^H(\mathbf{n})\Phi(\mathbf{n})\mathbf{H}_M(\mathbf{n})\right)^{-1} \\ &\quad H_M^H(n)\Phi^H(n)\Phi(n)\mathbf{d}_{f+1}(\mathbf{n}).\end{aligned} \quad (7)$$

We use the following relation,  $\Phi^H(n)\Phi(n) = I - \mathbf{c}(\mathbf{n})\mathbf{c}^H(\mathbf{n})$ , where  $\mathbf{c}(\mathbf{n}) \triangleq \sqrt{1-a^2} [a^n \ a^{n-1} \ \cdots \ a \ 1]^T$ . If we put the relation into (7), we can get a useful equation,

$$\bar{\mathbf{e}}_{p,f+1}(n) = \Phi(n)\mathbf{e}_{p,f}(\mathbf{n}) + \Phi(\mathbf{n})\hat{\mathbf{c}}_M(\mathbf{n})\zeta_M^{-c}(\mathbf{n})\mathbf{c}^H(\mathbf{n})\mathbf{e}_{p,f}(\mathbf{n}), \quad (8)$$

where  $\hat{\mathbf{c}}_M(n) = H_M(n)R_M^{-1}(n)H_M^H(n)\mathbf{c}(\mathbf{n})$ , and  $\zeta_M^c(n) = 1 - \mathbf{c}^H(\mathbf{n})\hat{\mathbf{c}}_M(\mathbf{n})$ . The last element of  $\bar{\mathbf{e}}_{p,f+1}(n)$  can be simply rewritten shown at 9-th line in Table 2.1., where  $\phi(n) = a^{-1}\sqrt{1-a^2}\mathbf{c}^H(\mathbf{n}) + [0 \ 0 \ \cdots \ -1/a]$ , and  $\tilde{\mathbf{c}}_M(n) = \mathbf{c}(\mathbf{n}) - \hat{\mathbf{c}}_M(\mathbf{n})$ . In the sequel, we have obtained two recursions for the forward smoothing error as shown in Table 2.1. Similarly to the forward smoothing error, the order-update recursion for the backward smoothing error (2) is obtained as

$$\mathbf{e}_{p+1,f}(\mathbf{n}) = \mathbf{e}_{p,f}(\mathbf{n}) - \mathbf{b}_M^H(\mathbf{n})\mathbf{d}_{f+1}(n)\xi_M^{-b}(\mathbf{n})\mathbf{b}_M(\mathbf{n}), \quad (9)$$

where  $\mathbf{b}_M(\mathbf{n})$  is the backward prediction error and  $\xi_M^b(n) = \mathbf{b}_M^H(\mathbf{n})\mathbf{b}_M(\mathbf{n})$ . Now, we need a time-update recursion for  $\mathbf{c}^H(\mathbf{n})\mathbf{e}_{p,f}(\mathbf{n}) (\triangleq \chi_M(n))$  in (8), which is easily obtained by using the time-update relation of cross-correlation of two error vectors as appeared in [5],

$$\chi_M(n) = a\chi_M(n-1) + \tilde{c}_M^*(n)\gamma_M^{-1}(n)e_{p,f}(n). \quad (10)$$

where  $\gamma_M(n)$  is the well-known conversion factor. Using the proposed order-recursive Laguerre prediction algorithm [3], the LSORL Laguerre smoother was obtained as shown in Table 2.1, where the prediction part is omitted in this paper by the page limit.

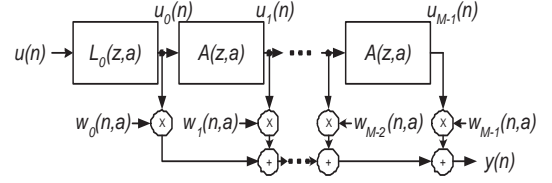


Fig. 1. Laguerre filter

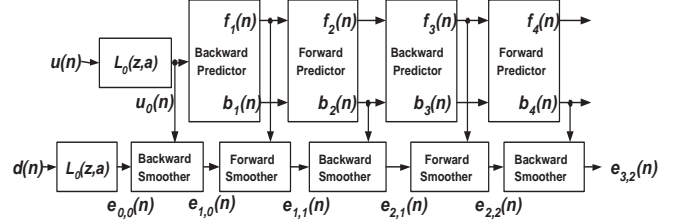


Fig. 2. The LSORL Laguerre smoother

## 2.1. Numerical examples and conclusion

Bernoulli sequence with  $\pm 1$  is generated and fed into an IIR-typed channel with

$$H(z) = \frac{(0.866z^{-1} - 0.866z^{-2} + 0.2165z^{-3})}{(1 + 0.3z^{-1} + 0.3z^{-2} + 0.5z^{-3})^{-1}}$$

and the SNR at the channel output is set to 30dB. The lattice realization of the smoother is chosen as so-called 'BBFBF...' form mentioned as shown in Figure 2; 'B' stands for the backward smoothing and 'F' stands for the forward smoothing. The tap-order of the LSORL Laguerre smoother is set to 6, and  $a$  is set to 0.5, which is obtained from a method proposed in [6]. For comparison, the 6-th order transversal LSORL smoother is used. Their MSE learning curves, which are averaged for 500 experiments, are depicted in Figure 3. This figure shows that the LSORL Laguerre smoother outperforms the transversal LSORL smoother for the equalization of scattering channels.

In this paper, we introduced the LSORL Laguerre smoother. For equalizing a channel having a long and oscillatory impulse response, it showed better performance than its transversal counterpart.

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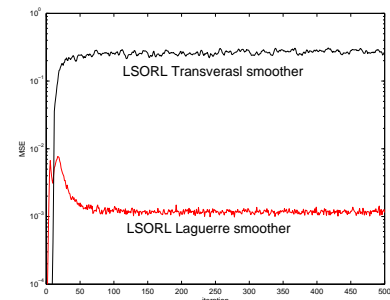


Fig. 3. Learning curves

Table 1. The LSORL Laguerre smoother

For $n=0, \dots$
$\sigma_m(-1) = \varphi_m(-1) = \chi_m(-1) = 0$ for all $m$
$e_{0,0}(n) = ae_{0,0}(n) + \sqrt{1-a^2}d(n)$
For $m=1, \dots, M-1$
LSORL Laguerre prediction: Omitted. [3]
LSORL Laguerre smoothing:
If $m$ is odd,
$\sigma_M(n) = \sigma_M(n-1) + \bar{e}_{p,f}^*(n)\bar{\gamma}_M^{-1}(n)f_M(n)$
$\bar{e}_{p,f+1}(n) = -(e_{p,f}(n) + \chi_M(n)\zeta_M^{-c}(n)\bar{e}_M(n))/a$
$e_{p,f+1}(n) = \bar{e}_{p,f+1} - \sigma_M^*(n)\xi_M^{-f}(n)f_M(n)$
If $m$ is odd,
$\varphi_M(n) = \varphi_M(n-1) + e_{p,f}^*(n)\gamma_M^{-1}(n)b_M(n)$
$e_{p+1,f}(n) = e_{p,f}(n) - \varphi_M^*(n)\xi_M^{-b}(n)b_M(n)$

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