Adaptive Immersion and Invariance Control of the Van der Pol Equation

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Abstract: We study the adaptive stabilization of the Van der Pol equation. A parameter update law is designed by the immersion and invariance method, and is used in conjunction with both the feedback linearization and backstepping control laws. Simulation results show that the responses obtained in the adaptive case are very similar to the known parameter case, and the parameter estimator converges to the true value.

Keywords: immersion and invariance control, adaptive nonlinear control, Van der Pol equation

1. Introduction

In [1], Astolfi and Ortega propose the immersion and invariance (I & I) method as a new tool to design a controller for nonlinear systems. This method is particularly useful when we know a stabilizing controller of a nominal reduced-order model and we would like to robustify it with respect to higher-order dynamics. A control law could be designed so that the full system dynamics is asymptotically immersed into the reduced-order one (the target system). The method can also be used in adaptive control problems, giving stabilizing control schemes that counter the effect of the uncertain parameters. The procedure does not invoke certainty equivalence, nor requires a linear parameterization. They apply the technique to design a stabilizing controller for a magnetic levitation system, a global tracking controller for a flexible joint robot, and an adaptive controller for a visual servoing system. Besides these works, there are currently scarcely any applications of the immersion and invariance technique to other nonlinear systems.

In this paper, we present an adaptive stabilizing controller design based on the immersion and invariance method for the Van der Pol equation. The control objective is to make origin globally asymptotically stable despite an unknown parameter in the system.

2. The Immersion and Invariance Method

Main results about the immersion and invariance technique can be summarized in the following theorems [1].

Theorem 1 Consider a nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let $x_* \in \mathbb{R}^n$ be the equilibrium point to be stabilized and let $p < n$.

Suppose we can find mappings

$$\alpha(\cdot) : \mathbb{R}^p \to \mathbb{R}^p \qquad \pi(\cdot) : \mathbb{R}^p \to \mathbb{R}^n \qquad c(\cdot) : \mathbb{R}^p \to \mathbb{R}^m$$
$$\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n-p} \qquad \psi(\cdot,\cdot) : \mathbb{R}^{n \times (n-p)} \to \mathbb{R}^m$$

such that the following conditions hold.

(A1) *(Target system)* The system

$$\dot{\xi} = \alpha(\xi) \tag{2}$$

with state $\xi \in \mathbb{R}^p$, has a globally asymptotically stable equilibrium at $\xi_* \in \mathbb{R}^p$ and $x_* = \pi(\xi_*)$.

(A2) *(Immersion condition)* For all $\xi \in \mathbb{R}^p$,

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi) \tag{3}$$

(A3) *(Implicit manifold)* The following set identity holds

$$\begin{aligned} &\{x \in \mathbb{R}^n \mid \phi(x) = 0\} \\ &= \{x \in \mathbb{R}^n \mid x = \pi(\xi) \text{ for some } \xi \in \mathbb{R}^p\} \end{aligned} \tag{4}$$

(A4) *(Manifold attractivity and trajectory boundedness)* All trajectories of the system

$$\dot{z} = \frac{\partial \phi}{\partial x}[f(x) + g(x)\psi(x, z)] \tag{5}$$
$$\dot{x} = f(x) + g(x)\psi(x, z) \tag{6}$$

are bounded and satisfy

$$\lim_{t \to \infty} z(t) = 0 \tag{7}$$

Then, x_* is a globally asymptotically stable equilibrium point of the closed-loop system

$$\dot{x} = f(x) + g(x)\psi(x, \phi(x))$$

In this case, we say that the system (1) is *I & I stabilizable* with respect to the target dynamics (2)

The immersion and invariance method can be extended to the problem of adaptive stabilization of nonlinear systems under the following assumption.

(A5) *(Stabilizability)* There exists a parameterized function $\Psi(x, \theta)$, where $\theta \in \mathbb{R}^q$, such that for some unknown $\theta_* \in \mathbb{R}^q$, the system

$$\dot{x} = f_*(x) := f(x) + g(x)\Psi(x, \theta_*) \tag{8}$$

has a globally asymptotically stable equilibrium at $x = x_*$.

The system (1) under the assumption (A5) is said to be *adaptively I & I stabilizable* if the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\Psi(x, \hat{\theta} + \beta_1(x)) \\ \dot{\hat{\theta}} &= \beta_2(x, \hat{\theta}) \end{aligned} \tag{9}$$

with extended state $(x, \hat{\theta})$ and the functions β_1 and β_2, is I & I stabilizable with target dynamics

$$\dot{\xi} = f_*(\xi). \tag{10}$$

Theorem 2 Consider the system (1) with assumptions (A5) and

(A6) *(Linearly parameterized control)* the function $\Psi(x, \theta)$ may be written as

$$\Psi(x, \theta) = \Psi_0(x) + \Psi_1(x)\theta \tag{11}$$

for some known functions $\Psi_0(x)$ and $\Psi_1(x)$.

Assume that there exists a function $\beta_1 : \mathbb{R}^n \to \mathbb{R}^m$ such that

(A7) *(Realizability)* $(\partial\beta_1/\partial x) f_*(x)$ is independent of the unknown parameters.

(A8) *(Manifold attractivity and trajectory boundedness)* All trajectories of the error system

$$\dot{x} = f_*(x) + g(x)\Psi_1(x)z \tag{12}$$

$$\dot{z} = \left[\frac{\partial\beta_1}{\partial x} g(x)\Psi_1(x)\right] z \tag{13}$$

are bounded and satisfy

$$\lim_{t\to\infty} g(x(t))\Psi_1(x(t))z(t) = 0.$$

Then, (1) is adaptively I & I stabilizable with the parameter update law given by

$$\beta_2(x) = -\frac{\partial\beta_1}{\partial x} f_*(x). \tag{14}$$

3. Van der Pol Equation

Consider the Van der Pol equation

$$\dot{x}_1 = x_2 \tag{15}$$

$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u \tag{16}$$

where ϵ is assumed to be an unknown parameter. The control objective is to make the origin globally asymptotically stable.

First, we assume that ϵ is known and design stabilizing control laws by the feedback linearization method [2] and the backstepping method [3].

By cancellation of the nonlinearity in the equation of $\dot{x}_2$, the feedback linearization control law is

$$u_{FL}(x, \epsilon) = -\epsilon(1 - x_1^2)x_2 + a_1 x_1 + a_2 x_2 \tag{17}$$

where $a_1 < 1$ and $a_2 < 0$ are design parameters.

The backstepping control law, on the other hand, does not try to make the system linear, but retains or introduces some useful nonlinearities while trying to stabilize the system. The design procedure is as follows:

Consider the scalar system

$$\dot{x}_1 = x_2 \tag{18}$$

Consider x_2 as the virtual control and choose

$$x_2 = \phi(x_1) = -x_1^3$$

The Lyapunov function for the subsystem (18) is $V = \frac{1}{2}x_1^2$. Then, we choose

$$-x_1 + \epsilon(1 - x_1^2)x_2 + u = -x_2\frac{\partial x_1^3}{\partial x_1} - x_1 - b_1(x_1^3 + x_2)$$

to stabilize the system with respect to the Lyapunov function $V_a = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1)^2$. Finally, the backstepping control law is

$$u_{BS}(x, \epsilon) = -\epsilon(1 - x_1^2)x_2 - 3x_1^2 x_2 - b_1(x_1^3 + x_2) \tag{19}$$

where $b_1 > 0$ is a design parameter.

Now, consider the case when ϵ is assumed to be unknown. To design a parameter update law by the immersion and invariance method, we select the target system as

$$\dot{x}_1 = x_2 \tag{20}$$

$$\dot{x}_2 = -x_1 + \theta_*(1 - x_1^2)x_2 + u(x, \theta_*) \tag{21}$$

where θ_* is the value of the unknown parameter ϵ to be estimated. The implicit manifold condition (A3) in this case is

$$\phi(x, \hat{\theta}) = \hat{\theta} - \theta_* + \beta_1(x) = 0$$

where $\hat{\theta}$ is the estimate of ϵ, and the off-the-manifold coordinate is

$$z = \hat{\theta} - \theta_* + \beta_1(x)$$

Its derivative is

$$\dot{z} = \beta_2(x) + \frac{\partial\beta_1}{\partial x}[f_0(x) + f_1(x)\theta_* + g(x)u]$$

where

$$f_0(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, f_1(x) = \begin{bmatrix} 0 \\ (1 - x_1^2)x_2 \end{bmatrix}, g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If we choose

$$\beta_2(x) = -\frac{\partial\beta_1}{\partial x}\left(f_0(x) + f_1(x)[\hat{\theta} + \beta_1(x)] + g(x)u\right) \tag{22}$$

and choose

$$\beta_1(x) = k(1 - x_1^2)\frac{x_2^2}{2} \tag{23}$$

where $k > 0$ is a design parameter, we obtain the parameter update law as

$$\begin{aligned}\dot{\hat{\theta}} &= \beta_2(x) \\ &= kx_1x_2^3 - k(1 - x_1^2) \\ &\quad \times [-x_1 + (1 - x_1^2)x_2(\hat{\theta} + \beta_1) + u]\end{aligned} \tag{24}$$

and the adaptive control law is

$$u = u_{FL}(x, \hat{\theta} + \beta_1) \tag{25}$$

in the case of the feedback linearization control (17), and

$$u = u_{BS}(x, \hat{\theta} + \beta_1) \tag{26}$$

in the case of the backstepping control (19).

The off-the-manifold dynamics becomes

$$\dot{z} = -[k(1 - x_1^2)^2 x_2^2]z \tag{27}$$

From (27), we see that the trajectory of z is bounded and converges to zero.

4. Simulation Results

Computer Simulations are performed with $\epsilon = 1$. The controller parameters for the feedback linearization control law (17) are chosen as $a_1 = -19$, $a_2 = -9$ and the controller parameters for the backstepping control law (19) is $b_1 = 5$. The parameter in the immersion and invariance parameter update law (24) is $k = 1$. The initial condition of the Van der Pol system is $x_1(0) = x_2(0) = 2.5$.

Figures 1 and 2 show the time response of the states x_1, x_2 under the feedback linearization control law when $\hat{\theta}(0) = 0.8$ and $\hat{\theta}(0) = 1.2$, respectively. The result of the fixed controller (17) (when ϵ is assumed to be known) is shown in thick line, while that of the adaptive controller (25) (when ϵ is assumed to be unknown and the parameter update law (24) is used) is shown in thin line. The value of the parameter estimate $\hat{\theta} + \beta_1$ is shown in Figure 3.

Similarly, Figures 4 and 5 show the time response of the states x_1, x_2 under the backstepping control law when $\hat{\theta}(0) = 0.8$ and $\hat{\theta}(0) = 1.2$, respectively. The result of the fixed controller (19) is shown in thick line, while that of the adaptive controller (26) is shown in thin line. The value of the parameter estimate $\hat{\theta} + \beta_1$ is also shown in Figure 6.

Figures 7 - 8 and 9 - 10 show the phase portrait of the system under the feedback linearization and backstepping control laws respectively. As in previous figures, the results of the fixed controllers are shown in thick line while those of the adaptive controllers are shown in thin line.

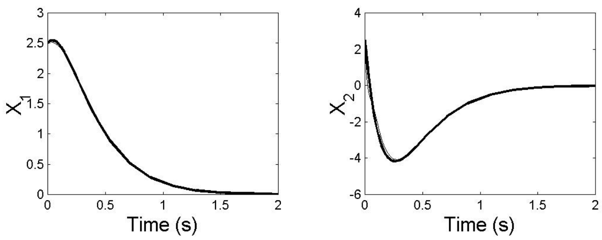

Fig. 1. Time responses of x_1 and x_2 under the feedback linearization control when $\hat{\theta}(0) = 0.8$

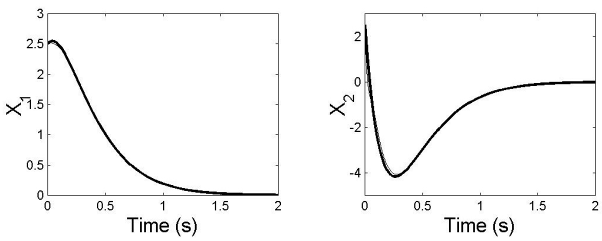

Fig. 2. Time responses of x_1 and x_2 under the feedback linearization control when $\hat{\theta}(0) = 1.2$

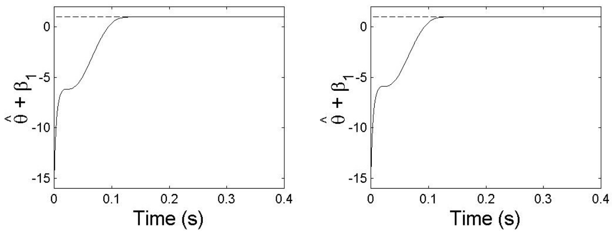

Fig. 3. The parameter estimate $\hat{\theta} + \beta_1$ under the feedback linearization control when (a) $\hat{\theta}(0) = 0.8$ and (b) $\hat{\theta}(0) = 1.2$

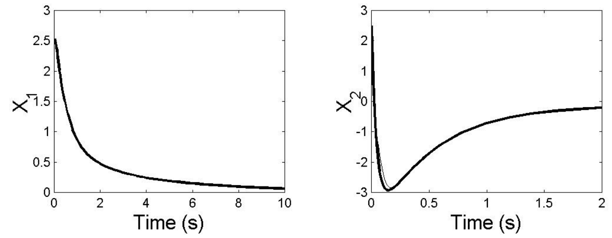

Fig. 4. Time responses of x_1 and x_2 under the backstepping control when $\hat{\theta}(0) = 0.8$

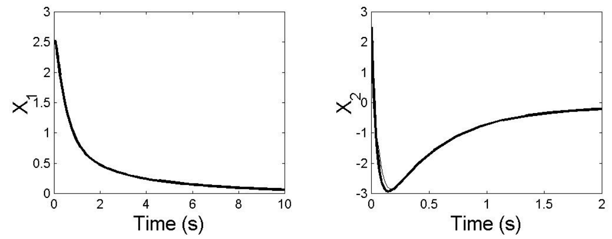

Fig. 5. Time responses of x_1 and x_2 under the backstepping control when $\hat{\theta}(0) = 1.2$

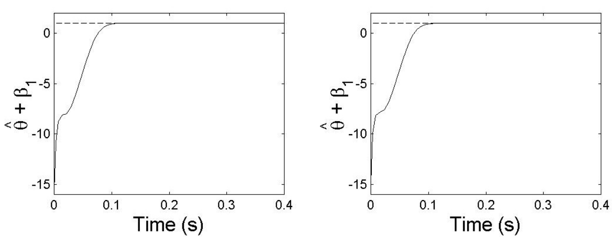

Fig. 6. The parameter estimate $\hat{\theta} + \beta_1$ under the backstepping control when (a) $\hat{\theta}(0) = 0.8$ and (b) $\hat{\theta}(0) = 1.2$

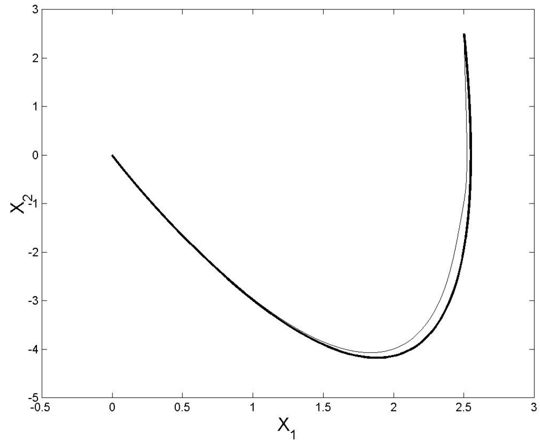

Fig. 7. Phase portrait of the system under the feedback linearization control with $\hat{\theta}(0) = 0.8$

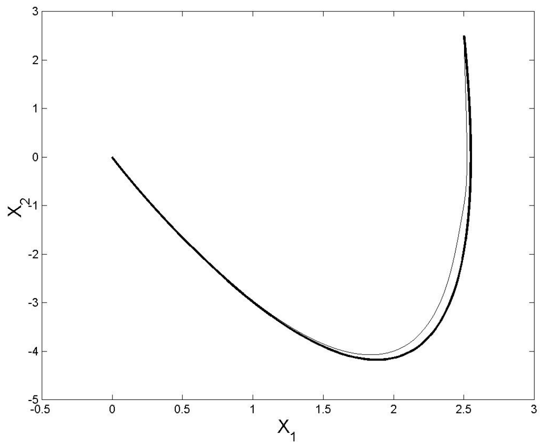

Fig. 8. Phase portrait of the system under the feedback linearization control with $\hat{\theta}(0) = 1.2$

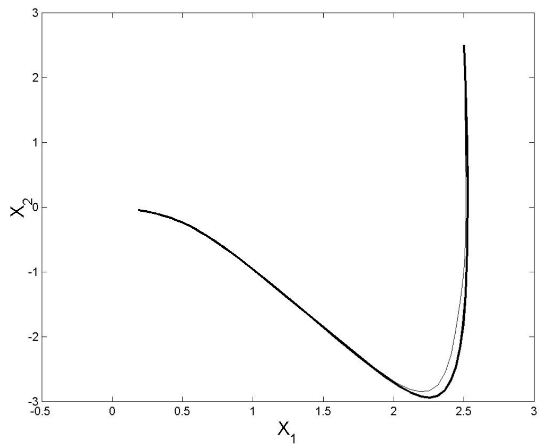

Fig. 9. Phase portrait of the system under the backstepping control with $\hat{\theta}(0) = 0.8$

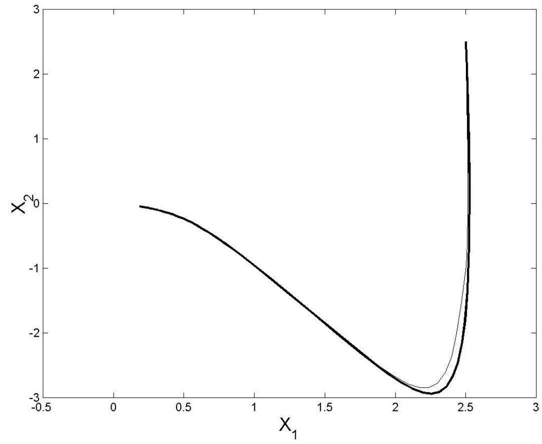

Fig. 10. Phase portrait of the system under the backstepping control with $\hat{\theta}(0) = 1.2$

It can be seen that the parameter update law designed by the immersion and invariance method can be used with both the feedback linearization control law and the backstepping control law such that the system responses do not differ much from the known parameter (fixed controller) case. The value of the parameter estimate also converges toward the true parameter value in both cases.

5. Conclusion

We design a parameter update law for the Van der Pol equation by the recently proposed immersion and invariance method. The parameter update law obtained can be used with both the feedback linearization control law and the backstepping control law to stabilize the system when a system parameter is unknown. The system responses of the adaptive controllers do not differ much from those of the fixed controllers and the value of the parameter estimate also converges toward the true parameter value in both cases.

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