Data-based Stability Analysis for MIMO Linear Time-invariant Discrete-time Systems

Un Sik Park and Masao Ikeda

Department of Computer-Controlled Mechanical Systems, Graduate School of Engineering
Osaka University, Suita, Osaka 565-0871, Japan
(Tel: +81-6-6879-7335; Fax: +81-6-6879-7335;
E-mail: aurapus@watt.mech.eng.osaka-u.ac.jp; ikeda@mech.eng.osaka-u.ac.jp)

Abstract: This paper presents a data-based stability analysis of a MIMO linear time-invariant discrete-time system, as an extension of the previous results for a SISO system. In the MIMO case, a similar discussion as in the case of a SISO system is also applied, except that an augmented input and output space is considered whose dimension is determined in relation to both the orders of the input and output vectors and the numbers of inputs and outputs. As certain subspaces of the input and output space, both output data space and closed-loop data space are defined, which contain all the behaviors of a system, respectively, with zero input in open-loop and with a control input in closed-loop. Then, we can derive the data-based stability conditions, in which the open-loop stability can be checked by using a data matrix whose column vectors span the output data space and the closed-loop stability can also be checked by using a data matrix whose column vectors span the closed-loop data space.

Keywords: data space approach, data-based stability analysis, MIMO, linear time-invariant discrete-time systems

1. Introduction

In the field of system and control theory, it has been long and widely accepted that the *model-based approach*—or equivalently the *parameter space approach*, which employs a mathematical model such as a transfer function, state equation, or kernel representation as the system representation, is the most effective, reliable, and well-established method for control system analysis and design. In this approach, the mathematical models can be obtained generally in two different ways: either by the analytical formulation from first principles, e.g., physical or chemical laws, or by the identification from the experimental input and output data.

The original motivation of the present study is as follows: When the input and output data is the only information available from a system, with no knowledge about the physical or chemical laws of its governing dynamics—which is common in practice, then there is no inevitability that a mathematical model must be identified from the input and output data to analyze or design a control system. In other words, the parameter space approach is not a unique method, and further it is possible that there exists another approach that does not need to involve any mathematical models. From this consideration, as an alternative to the parameter space approach, the authors have proposed the *data space approach* for stability analysis and control design of a linear time-invariant discrete-time system [1], [2].

In the data space approach, a set of the input and output data, which provides the necessary and sufficient information about the dynamic behavior of a system under certain condition, e.g., persistent excitation, is solely and directly used to analyze or design a control system. Furthermore, in this approach, it is possible to deal directly with the uncertainties in data, which is mainly caused by noise.

The basic idea of the data space approach was first introduced by the second author and his colleagues, and preliminary results have been reported in [3], [4]. In the recent papers, the authors have introduced the data-based stability conditions for a linear time-invariant single-input and single-output (SISO) discrete-time systems in both open-loop and closed-loop operations [1], [5]. In addition, by using the data-based stability condition for closed-loop systems, the data-based design method of stabilizing controllers has also been introduced in the framework of the data space approach [2], [6].

In the present paper, we will extend the previous results of the data-based stability analysis for a SISO system to the case of a linear time-invariant multiple-input and multiple-output (MIMO) discrete-time system. A similar discussion as in the case of a SISO system can also be applied to the MIMO case. First, we consider an *input and output space* as a linear vector space, whose dimension is determined in relation to both the numbers of inputs and outputs and the orders of the input vector and the output vector. In addition, the notions of output data space and closed-loop data space are defined as certain subspaces of the input and output space, which contain all the dynamic behaviors of a system, respectively, with zero input in open-loop and with a control input in closed-loop. Then, the stability of the system can be checked by examining how an arbitrarily given initial point behaves on the data space. In order that, in the framework of the Lyapunov's second method, we derive both open-loop and closed-loop versions of the data-based stability conditions, in which the open-loop stability can be checked by using a data matrix whose column vectors span the entire output data space, and the closed-loop stability can also be checked by using a data matrix whose column vectors span the entire closed-loop data space.

Throughout this paper, all the discussions are made upon the following assumptions:

- The system under consideration is a finite dimensional linear time-invariant discrete-time system.
- When the dynamics of the system considered in the above assumption is described as an input and output vector difference equation, the orders of the input vector and output vector are known in advance.
- all the input and output data are noise-free.

The following notations are used: for a subspace $\mathbb{S}$, $\mathbb{S}^{\perp}$ denotes the orthogonal complement of $\mathbb{S}$, and for a matrix M, $M^{\perp}$ denotes a matrix of full column rank such that $MM^{\perp} = 0$. $\mathbb{R}$ is the set of real numbers, and $\mathbb{N}$ is the set of non-negative integers. I_n denotes the n-dimensional identity matrix.

2. MIMO Linear Time-invariant Discrete-time Systems

In this section, we will examine how a dynamic behavior of a linear time-invariant discrete-time system with multiple outputs and multiple inputs can be described in the data space.

Let us first define the output vector $\mathbf{y}(k) \in \mathbb{R}^N$ and the input vector $\mathbf{u}(k) \in \mathbb{R}^M$, which consist of N outputs and M inputs respectively, as

$$\mathbf{y}(k) = \left[\begin{array}{cccc} y_1(k) & y_2(k) & \cdots & y_N(k) \end{array}\right]^T, \quad k \in \mathbb{N} \tag{1}$$

$$\mathbf{u}(k) = \left[\begin{array}{cccc} u_1(k) & u_2(k) & \cdots & u_M(k) \end{array}\right]^T, \quad k \in \mathbb{N}, \tag{2}$$

where $y_i(k)\,(i = 1,2,\cdots,N)$ denotes the k-th time instant data of the i-th output y_i, and $u_j(k)\,(j = 1,2,\cdots,M)$ denotes the k-th time instant data of the j-th input u_j. Then, we consider a bahavior of a linear time-invariant discrete-time system with N outputs and M inputs, which can be described by using an input and output vector difference equation as

$$\begin{aligned} &\mathbf{y}(k+n) + A_{n-1}\mathbf{y}(k+n-1) + \cdots + A_1\mathbf{y}(k+1) + A_0\mathbf{y}(k) \\ &= B_m\mathbf{u}(k+m) + \cdots + B_1\mathbf{u}(k+1) + B_0\mathbf{u}(k), \quad k \in \mathbb{N}, \end{aligned} \tag{3}$$

where $A_i \in \mathbb{R}^{N\times N}(i = 0,1,\cdots,n-1)$ and $B_j \in \mathbb{R}^{N\times M}(j = 0,1,\cdots,m)$ are constant matrices defined as

$$A_i = \begin{bmatrix} a_{i(1,1)} & \cdots & a_{i(1,N)} \\ \vdots & \ddots & \vdots \\ a_{i(N,1)} & \cdots & a_{i(N,N)} \end{bmatrix}, \quad i = 0,1,\cdots,n-1, \tag{4}$$

$$B_j = \begin{bmatrix} b_{j(1,1)} & \cdots & b_{j(1,M)} \\ \vdots & \ddots & \vdots \\ b_{j(N,1)} & \cdots & b_{j(N,M)} \end{bmatrix}, \quad j = 0,1,\cdots,m. \tag{5}$$

In (3), without loss of generality, the coefficient matrix of the highest order term of the output vector is assumed to be N-dimensional identity matrix. In addition, we assume that B_m is not a null matrix and both A_0 and B_0 are not null matrices at the same time, then n and m denote respectively the orders of the output vector and the input vector, with satisfying $n > m$.

Next, let us consider the *input and output space* $\mathcal{D}$, defined as

$$\mathcal{D} = \mathbb{R}^{(n+1)N+(m+1)M}, \tag{6}$$

where the dimension of the input and output space is denoted as $\sigma := \dim\mathcal{D} = (n+1)N + (m+1)M$. Here, if we introduce a set of data vectors $d(k) \in \mathcal{D}$, each of which consists of both $(n+1)$ consecutive output vectors and $(m+1)$ consecutive input vectors, defined as

$$d(k) \triangleq \left[\begin{array}{ccc:ccc} \mathbf{y}(k+n)^T & \cdots & \mathbf{y}(k)^T & \mathbf{u}(k+m)^T & \cdots & \mathbf{u}(k)^T \end{array}\right]^T, \; k \in \mathbb{N}, \tag{7}$$

then the input and output vector difference equation in (3) can be rewritten as

$$\Theta^T d(k) = 0, \quad k \in \mathbb{N}, \tag{8}$$

where

$$\Theta = \left[\begin{array}{cccc:ccc} I_N & A_{n-1}^T & \cdots & A_0^T & -B_m^T & \cdots & -B_0^T \end{array}\right]^T \in \mathbb{R}^{\sigma\times N}. \tag{9}$$

From (8), it is easily seen that the degrees of freedom of the data vector $d(k)$ are restricted by N independent constraints imposed by Θ. Therefore, all the data vectors $d(k), k \in \mathbb{N}$ satisfying (8), i.e., all the behaviors of (3), have to reside in a certain subspace of $\mathcal{D}$, defined as

$$\mathcal{D}_o \triangleq \{\, d \in \mathcal{D} \mid d \perp (\operatorname{span}\{\Theta\}) \,\}, \tag{10}$$

where the dimension of $\mathcal{D}_o$ is denoted as $\sigma_o (= \dim\mathcal{D}_o = \sigma - N = nN + (m+1)M)$. We call $\mathcal{D}_o$ the *open-loop data space*, which can serve as the system representation in the data space.

3. Data-based Stability Condition for Open-loop Systems

In this section, we derive a stability condition that enables us to test the open-loop stability of a MIMO system by using a set of its output data only.

3.1. Output Data Space

First, let us consider a behavior described by an output vector difference equation as

$$\mathbf{y}(k+n) + A_{n-1}\mathbf{y}(k+n-1) + \cdots + A_0\mathbf{y}(k) = 0, \quad k \in \mathbb{N}, \tag{11}$$

which corresponds to the zero-input dynamics of the system as in (3). To examine the behavior of (11), let us introduce a set of *output data vectors* $d_y(k) \in \mathbb{R}^{(n+1)N}$, each of which consists of $(n+1)$ consecutive output vectors, as

$$d_y(k) \triangleq \left[\begin{array}{cccc} \mathbf{y}(k+n)^T & \mathbf{y}(k+n-1)^T & \cdots & \mathbf{y}(k)^T \end{array}\right]^T, \quad k \in \mathbb{N}. \tag{12}$$

Then, the output vector difference equation in (11) can be rewritten as

$$\Theta_y^T d_y(k) = 0, \quad k \in \mathbb{N}, \tag{13}$$

where

$$\Theta_y = \left[\begin{array}{cccc} I_N & A_{n-1}^T & \cdots & A_0^T \end{array}\right]^T \in \mathbb{R}^{(n+1)N\times N}. \tag{14}$$

From (13), it can be easily seen that the degrees of freedom of the output data vector $d_y(k)$ are restricted by N independent constraints imposed by Θ_y. Therefore, all the output data vectors $d_y(k), k \in \mathbb{N}$ satisfying (13), i.e., all the behaviors of (11), have to reside in a certain subspace, called the *output data space*, as

$$\mathcal{D}_y \triangleq \{\, d_y \in \mathbb{R}^{(n+1)N} \mid d_y \perp (\operatorname{span}\{\Theta_y\}) \,\}, \tag{15}$$

where the dimension of $\mathcal{D}_y$ is denoted as $\sigma_y (= \dim\mathcal{D}_y = (n+1)N - N = nN)$.

Next, let us consider a set of time instants as

$$\mathcal{K}_y = \{\, k_1, k_2, \cdots, k_{\sigma_y} \,\} \subset \mathbb{N}, \tag{16}$$

whose elements indicate any σ_y time instants that need not necessarily be consecutive. Then, we introduce an *output data matrix* $\Psi_y(\mathcal{K}_y) \in \mathbb{R}^{(n+1)N\times\sigma_y}$, whose columns are σ_y output data vectors of (12) at the time instants in $\mathcal{K}_y$, as

$$\Psi_y(\mathcal{K}_y) \triangleq \left[\begin{array}{cccc} d_y(k_1) & d_y(k_2) & \cdots & d_y(k_{\sigma_y}) \end{array}\right]. \tag{17}$$

Now, let us suppose that, for an arbitrary set $\mathcal{K}_y$, the column vectors of $\Psi_y(\mathcal{K}_y)$ are linearly independent or, equivalently, span the whole output data space, i.e., $\operatorname{span}\{\Psi_y(\mathcal{K}_y)\} = \mathcal{D}_y$. Then, the set of the column vectors of $\Psi_y(\mathcal{K}_y)$ forms a basis of $\mathcal{D}_y$, hence we call $\Psi_y(\mathcal{K}_y)$ as a basis matrix of the output data space $\mathcal{D}_y$ and denote it as Ψ_y. Since the basis matrix Ψ_y contains the necessary and sufficient information about the dynamic behavior of (11), it is possible to investigate the system's characteristics such as open-loop stability by using Ψ_y.

3.2. Data-based Open-loop Stability Condition

To see more effectively how the output data vector of (12) behaves in the output data space, we consider an equivalent system representation to (11) as

$$\begin{aligned} x(k+1) &= Ax(k) \\ \mathbf{y}(k) &= Cx(k) \end{aligned}, \quad k \in \mathbb{N}, \tag{18}$$

where the vector $x(k) \in \mathbb{R}^{\sigma_y}$ corresponds to n consecutive output vectors as

$$x(k) = \begin{bmatrix} \mathbf{y}(k+n-1)^T & \mathbf{y}(k+n-2)^T & \cdots & \mathbf{y}(k)^T \end{bmatrix}^T, \tag{19}$$

and the matrices $A \in \mathbb{R}^{\sigma_y \times \sigma_y}$ and $C \in \mathbb{R}^{N \times \sigma_y}$ are given as

$$A = \begin{bmatrix} A_{n-1} & \cdots & A_1 & A_0 \\ I_N & & & 0 \\ & \ddots & & \vdots \\ & & I_N & 0 \end{bmatrix} \tag{20}$$

$$C = \begin{bmatrix} 0 & \cdots & 0 & I_N \end{bmatrix}. \tag{21}$$

Now, let us consider a data matrix $\Phi(\mathcal{K}_y) \in \mathbb{R}^{\sigma_y \times \sigma_y}$, which consists of σ_y vectors x at the time instants in $\mathcal{K}_y$, as

$$\Phi(\mathcal{K}_y) = \begin{bmatrix} x(k_1) & x(k_2) & \cdots & x(k_{\sigma_y}) \end{bmatrix}. \tag{22}$$

In addition, for another set of time instants, which is one time-step forward-shifted version of the set $\mathcal{K}_y$, as

$$\mathcal{K}_y^{+1} = \{ k_1+1, k_2+1, \cdots, k_{\sigma_y}+1 \} \subset \mathbb{N}, \tag{23}$$

let us consider a data matrix $\Phi(\mathcal{K}_y^{+1}) \in \mathbb{R}^{\sigma_y \times \sigma_y}$ as

$$\Phi(\mathcal{K}_y^{+1}) = \begin{bmatrix} x(k_1+1) & x(k_2+1) & \cdots & x(k_{\sigma_y}+1) \end{bmatrix}. \tag{24}$$

Then, another expression for the constraints on the output data vectors $\mathbf{y}(k), k \in \mathbb{N}$ is given as

$$\begin{bmatrix} I_{nN} & -A \end{bmatrix} \begin{bmatrix} \Phi(\mathcal{K}_y^{+1}) \\ \Phi(\mathcal{K}_y) \end{bmatrix} = 0. \tag{25}$$

From this, we have the relationship between the data matrices $\begin{bmatrix} \Phi(\mathcal{K}_y^{+1})^T & \Phi(\mathcal{K}_y)^T \end{bmatrix}^T$ and the output data matrix $\Psi_y(\mathcal{K}_y)$ in (17) given as

$$\begin{bmatrix} \Phi(\mathcal{K}_y^{+1}) \\ \Phi(\mathcal{K}_y) \end{bmatrix} = \left[\begin{array}{c:c} I_{nN} & 0 \\ \hdashline 0 & I_{nN} \end{array}\right] \Psi_y(\mathcal{K}_y), \tag{26}$$

where 0 denotes an $nN \times N$ null matrix.

From these settings, we provide a data-based open-loop stability condition for a MIMO linear time-invariant discrete-times system as follows:

Theorem 1. *The following statements are equivalent.*

1) *The system* (18) *has an asymptotically stable equilibrium* $x = 0$.

2) *For all the behaviors* $x(k) \neq 0, k \in \mathbb{N}$ *that satisfy* (18), *there exists a positive definite symmetric matrix* $P \in \mathbb{R}^{\sigma_y \times \sigma_y}$ *such that*

$$x^T(k+1)Px(k+1) - x^T(k)Px(k) < 0. \tag{27}$$

3) *For a basis matrix* Ψ_y *of the output data space* $\mathcal{D}_y$, *there exists a positive definite symmetric matrix* $P \in \mathbb{R}^{\sigma_y \times \sigma_y}$ *such that*

$$\Psi_y^T \left[\begin{array}{c:c} I_{nN} & 0 \\ \hdashline 0 & I_{nN} \end{array}\right]^T \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \left[\begin{array}{c:c} I_{nN} & 0 \\ \hdashline 0 & I_{nN} \end{array}\right] \Psi_y < 0. \tag{28}$$

Proof. Since 1) $\Leftrightarrow$ 2) is obvious, only 2) $\Leftrightarrow$ 3) is shown here. First, (27) can be rewritten as

$$\begin{bmatrix} x^T(k+1) & x^T(k) \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix} < 0. \tag{29}$$

Then, the condition 2) is equivalent to that, for all the behaviors $\begin{bmatrix} x^T(k+1) & x^T(k) \end{bmatrix}^T \neq 0, k \in \mathbb{N}$ that satisfy

$$\begin{bmatrix} I_{nN} & -A \end{bmatrix} \begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix} = 0, \quad k \in \mathbb{N}, \tag{30}$$

there exists a positive definite symmetric matrix P that satisfies (29). Furthermore, from (29) and (30), an equivalent condition is obtained by Finsler's lemma [7] as

$$\begin{bmatrix} I_{nN} & -A \end{bmatrix}^{\perp T} \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I_{nN} & -A \end{bmatrix}^{\perp} \tag{31}$$

Here, since the matrix $\begin{bmatrix} I_{nN} & -A \end{bmatrix} \in \mathbb{R}^{\sigma_y \times 2\sigma_y}$ has the rank of σ_y, the rank of $\begin{bmatrix} I_{nN} & -A \end{bmatrix}^{\perp}$ is σ_y. Hence, from (25) and (26), the following relation holds:

$$\mathrm{span}\left\{ \begin{bmatrix} I_{nN} & -A \end{bmatrix}^{\perp} \right\} = \mathrm{span}\left\{ \left[\begin{array}{c:c} I_{nN} & 0 \\ \hdashline 0 & I_{nN} \end{array}\right] \Psi_y \right\} \tag{32}$$

As a consequence, (31) and (28) are equivalent, hence 2) $\Leftrightarrow$ 3) is proved. □

In Theorem 1, the condition 3) is one of the main results of this paper, which enables us to check the open-loop stability of a MIMO linear time-invariant discrete-time system—without involving any information on the parameters of A—by using only a basis matrix Ψ_y of the output data space $\mathcal{D}_y$, i.e., a set of the experimental output data.

From now on, let us briefly examine how we can construct a basis matrix Ψ_y from a set of the output data, which is generated from experiments as a zero-input response of the system. In a MIMO system, unlike the case of a SISO system, it is not always possible to obtain the set of the output data, which has enough degrees of freedom to construct a basis matrix Ψ_y, i.e., a set of σ_y linearly independent output data vectors, from the experiments for a single well-chosen initial condition. However, in general case, we can obtain an extensive set of the output data by operating multiple experiments for σ_y linearly independent initial conditions.

In order that, let us consider a set of σ_y linearly independent initial conditions as

$$\mathcal{X}_0 = \{ x_1(0), \; x_2(0), \; \cdots, \; x_{\sigma_y}(0) \}. \tag{33}$$

Now, if we operate σ_y-time experiments with respect to each initial conditions in $\mathcal{X}_0$, then a set of data vectors is obtained as follows:

$$\Big\{ x_1(0), Ax_1(0), \cdots, A^{r_1-1}x_1(0) \,\vdots\, x_2(0), Ax_2(0), \cdots, A^{r_2-1}x_2(0) \,\vdots\, \cdots \,\vdots\, x_{\sigma_y}(0), \; Ax_{\sigma_y}(0), \; \cdots, \; A^{r_{\sigma_y}-1}x_{\sigma_y}(0) \Big\}, \tag{34}$$

where r_i $(i = 1, 2, \cdots, \sigma_y)$ denotes the number of linearly independent data vectors that can be obtained with respect to each initial conditions $x_i(0)$ $(i = 1, 2, \cdots, \sigma_y)$. Since, in the set of data vectors in (34), there exist σ_y linearly independent data vectors except for such initial data vectors $x_i(0)$ $(i = 1, 2, \cdots, \sigma_y)$, it is possible to obtain a basis matrix Ψ_y.

4. Data-based Stability Condition for Closed-loop Systems

In this section, we derive a stability condition that enables us to test the closed-loop stability of a controlled MIMO system by using a set of its input and output data only.

4.1. Closed-loop Data Space

Let us first reconsider the behavior of a MIMO linear time-invariant discrete-time system to be controlled as

$$\begin{aligned}&\hat{\mathbf{y}}(k+n)+A_{n-1}\hat{\mathbf{y}}(k+n-1)+\cdots+A_1\hat{\mathbf{y}}(k+1)+A_0\hat{\mathbf{y}}(k)\\&=B_m\hat{\mathbf{u}}(k+m)+\cdots+B_1\hat{\mathbf{u}}(k+1)+B_0\hat{\mathbf{u}}(k),\quad k\in\mathbb{N}.\end{aligned}\tag{35}$$

Next, we also consider a dynamic controller, which feeds the outputs of (35) back to generate the control input, as

$$\begin{aligned}&\hat{\mathbf{u}}(k+m)+D_{m-1}\hat{\mathbf{u}}(k+m-1)+\cdots+D_1\hat{\mathbf{u}}(k+1)+D_0\hat{\mathbf{u}}(k)\\&=C_{n-1}\hat{\mathbf{y}}(k+n-1)+\cdots+C_1\hat{\mathbf{y}}(k+1)+C_0\hat{\mathbf{y}}(k),\quad k\in\mathbb{N},\end{aligned}\tag{36}$$

where the coefficient matrices $C_i\in\mathbb{R}^{M\times N}\,(i=0,1,\cdots,n-1)$ and $D_j\in\mathbb{R}^{M\times M}\,(j=0,1,\cdots,m-1)$ are constant matrices defined as

$$C_i=\begin{bmatrix}c_{i(1,1)} & \cdots & c_{i(1,N)}\\ \vdots & & \vdots\\ c_{i(M,1)} & \cdots & c_{i(M,N)}\end{bmatrix},\quad i=0,1,\cdots,n-1,\tag{37}$$

$$D_j=\begin{bmatrix}d_{j(1,1)} & \cdots & d_{j(1,M)}\\ \vdots & & \vdots\\ d_{j(M,1)} & \cdots & d_{j(M,M)}\end{bmatrix},\quad j=0,1,\cdots,m-1.\tag{38}$$

Here, $C_i\,(i=0,1,\cdots,n-1)$ and $D_j\,(j=0,1,\cdots,m-1)$ are assumed to be given for stability test. Note also that $\hat{\mathbf{y}}(k)\in\mathbb{R}^N$ and $\hat{\mathbf{u}}(k)\in\mathbb{R}^M$ are introduced to denote the closed-loop output and input vectors respectively, which are generated by satisfying (35) and (36) simultaneously, in order to distinguish them from the open-loop output and input vectors $\mathbf{y}(k)$, $\mathbf{u}(k)$.

Now, if we consider a set of *closed-loop data vectors* $\hat{d}(k)\in\mathcal{D}$, each of which consists of both $(n+1)$ consecutive closed-loop output vectors and $(m+1)$ consecutive closed-loop input vectors, defined as

$$\hat{d}(k)\triangleq\left[\hat{\mathbf{y}}(k+n)^T\ \cdots\ \hat{\mathbf{y}}(k)^T\ \vdots\ \hat{\mathbf{u}}(k+m)^T\ \cdots\ \hat{\mathbf{u}}(k)^T\right]^T,\ k\in\mathbb{N},\tag{39}$$

then the input and output vector difference equations in (35) and (36) can be rewritten as

$$\begin{bmatrix}\Theta^T\\ \Theta_c^T\end{bmatrix}\hat{d}(k)=0,\quad k\in\mathbb{N},\tag{40}$$

where Θ is given as in (9) and

$$\Theta_c=\left[\ 0\ \ -C_{n-1}^T\ \cdots\ -C_0^T\ \vdots\ I_N\ \ D_{m-1}^T\ \cdots\ D_0^T\ \right]^T\in\mathbb{R}^{\sigma\times M}.\tag{41}$$

From this, it is easily seen that the degrees of freedom of the closed-loop data vectors $\hat{d}(k)$ are restricted by $(N+M)$ independent constraints imposed by Θ and Θ_c. Therefore, all the closed-loop data vectors $\hat{d}(k)$, $k\in\mathbb{N}$ satisfying (40), i.e., all the behaviors of the closed-loop system ((35) and (36)), have to reside in a certain subspace of $\mathcal{D}$, called the *closed-loop data space*, as

$$\mathcal{D}_c\triangleq\{\hat{d}\in\mathcal{D}\mid\hat{d}\perp(\operatorname{span}\{[\Theta\ \Theta_c]\})\}.\tag{42}$$

where the dimension of $\mathcal{D}_c$ is denoted as $\sigma_c(=\dim\mathcal{D}_c=\sigma-N-M=nN+nM)$.

Next, let us consider a set of time instants as

$$\mathcal{K}_c=\{k_1,k_2,\cdots,k_{\sigma_c}\}\subset\mathbb{N},\tag{43}$$

whose elements indicate any σ_c time instants that need not necessarily be consecutive. Then, we introduce a *closed-loop data matrix* $\Psi_c(\mathcal{K}_c)\in\mathbb{R}^{\sigma\times\sigma_c}$, whose columns are σ_c closed-loop data vectors of (39) at the time instants in $\mathcal{K}_c$, as

$$\Psi_c(\mathcal{K}_c)\triangleq\left[\ \hat{d}(k_1)\ \ \hat{d}(k_2)\ \cdots\ \hat{d}(k_{\sigma_c})\ \right].\tag{44}$$

Now, let us suppose that, for an arbitrary set $\mathcal{K}_c$, the column vectors of $\Psi_c(\mathcal{K}_c)$ are obtained to be linearly independent or, equivalently, span the whole closed-loop data space $\mathcal{D}_c$. Then, the set of the column vectors of $\Psi_c(\mathcal{K}_c)$ forms a basis of $\mathcal{D}_c$. Therefore, we call such data matrix $\Psi_c(\mathcal{K}_c)$ as a basis matrix of the closed-loop data space $\mathcal{D}_c$ and denote it as Ψ_c. Similarly to the open-loop case, since the basis matrix Ψ_c contains the necessary and sufficient information about the dynamic behavior of the closed-loop system consists of (35) and (36), it is possible to investigate the closed-loop stability by using Ψ_c.

4.2. Data-based Closed-loop Stability Condition

To see how the closed-loop input and output vectors, which satisfy (35) and (36) simultaneously, behave in the closed-loop data space $\mathcal{D}_c$ as in (42), we consider an equivalent system representation to (35) and (36) as

$$\left.\begin{aligned}&E_c x_c(k+1)=A_c x_c(k)\\&\begin{bmatrix}\hat{\mathbf{y}}(k)\\ \hat{\mathbf{u}}(k)\end{bmatrix}=C_c x_c(k)\end{aligned}\right\},\quad k\in\mathbb{N},\tag{45}$$

where $x_c(k)\in\mathbb{R}^{\sigma_c}$ denotes a vector, which consists of both n consecutive closed-loop output vectors and m consecutive closed-loop input vectors, as

$$\begin{aligned}x_c(k)=\big[&\ \hat{\mathbf{y}}(k+n-1)^T\ \ \hat{\mathbf{y}}(k+n-2)^T\ \cdots\ \hat{\mathbf{y}}(k)^T\ \vdots\\&\hat{\mathbf{u}}(k+m-1)^T\ \ \hat{\mathbf{u}}(k+m-2)^T\ \cdots\ \hat{\mathbf{u}}(k)^T\ \big]^T,\end{aligned}\tag{46}$$

and the matrices $E_c, A_c\in\mathbb{R}^{\sigma_c\times\sigma_c}$ and $C_c\in\mathbb{R}^{(N+M)\times\sigma_c}$ are given as

$$E_c=\left[\begin{array}{ccc|ccc}I_N & & & B_m & 0\ \cdots & 0\\ & \ddots & & & [0] & \\ & & I_N & & & \\ \hline & & & I_N & & \\ & [0] & & & \ddots & \\ & & & & & I_N\end{array}\right]\tag{47}$$

$$A_c=\left[\begin{array}{cccc|cccc}-A_{n-1} & \cdots & -A_1 & -A_0 & B_{m-1} & \cdots & B_1 & B_0\\ I_N & & & 0 & & & & \\ & \ddots & & \vdots & & [0] & & \\ & & I_N & 0 & & & & \\ \hline C_{n-1} & \cdots & C_1 & C_0 & -D_{m-1} & \cdots & -D_1 & -D_0\\ & & & & I_N & & & 0\\ & [0] & & & & \ddots & & \vdots\\ & & & & & & I_N & 0\end{array}\right]\tag{48}$$

$$C_c = \left[\begin{array}{ccc:ccc} 0 & \quad & 0 \quad I_N & 0 & \quad & 0 \quad 0 \\ 0 & \quad & 0 \quad 0 & 0 & \quad & 0 \quad I_N \end{array}\right]. \tag{49}$$

Now, let us consider a data matrix $\Phi_c(\mathcal{K}_c) \quad \mathbb{R}^{\sigma_c \quad \sigma_c}$, which consists of σ_c vectors x_c at the time instants in $\mathcal{K}_c$, as

$$\Phi_c(\mathcal{K}_c) = \left[\begin{array}{cccc} x_c(k_1) & x_c(k_2) & \quad & x_c(k_{\sigma_c}) \end{array}\right]. \tag{50}$$

In addition, for another set of time instants, which is one time-step forward shifted version of $\mathcal{K}_c$, as

$$\mathcal{K}_c^{+1} = \quad k_1+1,\ k_2+1, \quad ,\ k_{\sigma_c}+1 \quad \mathbb{N}, \tag{51}$$

let us consider a data matrix $\Phi_c(\mathcal{K}_c^{+1}) \quad \mathbb{R}^{\sigma_c \quad \sigma_c}$ as

$$\Phi_c(\mathcal{K}_c^{+1}) = \left[\begin{array}{cccc} x_c(k_1+1) & x_c(k_2+1) & \quad & x_c(k_{\sigma_c}+1) \end{array}\right]. \tag{52}$$

Then, another expression for the constraints on the closed-loop output and input data vectors $\hat{\mathbf{y}}(k), \hat{\mathbf{u}}(k),\ k \quad \mathbb{N}$ is given as

$$\left[\begin{array}{cc} E_c & A_c \end{array}\right] \left[\begin{array}{c} \Phi_c(\mathcal{K}_c^{+1}) \\ \Phi_c(\mathcal{K}_c) \end{array}\right] = 0. \tag{53}$$

From this, we have the relationship between the data matrices $[\ \Phi_c(\mathcal{K}_c^{+1})^T \quad \Phi_c(\mathcal{K}_c)^T\ ]^T$ and the closed-loop data matrix $\Psi_c(\mathcal{K}_c)$ in (44) given as

$$\left[\begin{array}{c} \Phi_c(\mathcal{K}_c^{+1}) \\ \Phi_c(\mathcal{K}_c) \end{array}\right] = \left[\begin{array}{c:c:c:c} I_{nN} & 0 & \multicolumn{2}{c}{0} \\ \hdashline \multicolumn{2}{c:}{0} & I_{mM} & 0 \\ \hdashline 0 & I_{nN} & \multicolumn{2}{c}{0} \\ \hdashline \multicolumn{2}{c:}{0} & 0 & I_{mM} \end{array}\right] \Psi_c(\mathcal{K}_c), \tag{54}$$

where 0 in the same block with I_{nN} denotes an $nN \quad N$ null matrix and also 0 in the same block with I_{mM} denotes an $mM \quad M$ null matrix.

From these settings, we provide a data-based closed-loop stability condition for a MIMO linear time-invariant discrete-times system as follows:

Theorem 2. *The following statements are equivalent.*

1) *The system* (45) *has an asymptotically stable equilibrium* $x_c = 0$.
2) *For all the behaviors* $x_c(k) = 0,\ k \quad \mathbb{N}$ *that satisfy* (45), *there exists a positive definite symmetric matrix* $P \quad \mathbb{R}^{\sigma_c \quad \sigma_c}$ *such that*

$$x_c^T(k+1)Px_c(k+1) \quad x_c^T(k)Px_c(k) < 0. \tag{55}$$

3) *For a basis matrix* Ψ_c *of the closed-loop data space* $\mathcal{D}_c$, *there exists a positive definite symmetric matrix* $P \quad \mathbb{R}^{\sigma_c \quad \sigma_c}$ *such that*

$$\Psi_c^T \left[\begin{array}{c:c:c:c} I_{nN} & 0 & \multicolumn{2}{c}{0} \\ \hdashline \multicolumn{2}{c:}{0} & I_{mM} & 0 \\ \hdashline 0 & I_{nN} & \multicolumn{2}{c}{0} \\ \hdashline \multicolumn{2}{c:}{0} & 0 & I_{mM} \end{array}\right]^T \left[\begin{array}{cc} P & 0 \\ 0 & P \end{array}\right] \left[\begin{array}{c:c:c:c} I_{nN} & 0 & \multicolumn{2}{c}{0} \\ \hdashline \multicolumn{2}{c:}{0} & I_{mM} & 0 \\ \hdashline 0 & I_{nN} & \multicolumn{2}{c}{0} \\ \hdashline \multicolumn{2}{c:}{0} & 0 & I_{mM} \end{array}\right] \Psi_c < 0. \tag{56}$$

Proof. Since the matrix E_c in (45) is nonsingular, we have

$$x_c(k+1) = E_c^{\ 1} A_c x_c(k). \tag{57}$$

Therefore, it can be proved in similar way to that of Theorem 1. □

The condition 3) in Theorem 2 is another main result of this paper, which enables us to check the closed-loop stability of a controlled MIMO system—without involving any information on the parameters of E_c and A_c— by using only a basis matrix Ψ_c of the closed-loop data space $\mathcal{D}_c$, i.e., a set of the closed-loop input and output vectors. Also, in a similar way to the open-loop case, it is possible to obtain a basis matrix Ψ_c.

5. Conclusions

In the present paper, as an extension of the previous results for a SISO system, we have developed both open-loop and closed-loop versions of the data-based stability conditions for a MIMO linear time-invariant discrete-time system, which do not need to employ any mathematical models.

Throughout the paper, a similar discussion as in the case of a SISO system is carried out. However, as a more general case of a MIMO system, we also need to pay attention to the case when the orders of the output vector and the input vector in each equations of a vector difference equation are not identical. This issue will be dealt with later in another paper.

References

[1] U. S. Park and M. Ikeda, "Data-based stability analysis for linear discrete-time systems," *Proceedings of the 43rd IEEE Conference on Decision and Control*, pp. 1721–1723, 2004.

[2] U. S. Park and M. Ikeda, "Data-based control for linear time-invariant discrete-time systems," *Proceedings of the 2004 International Conference on Control, Automation, and Systems*, pp. 1993–1998, 2004.

[3] M. Ikeda and N. Hayashi, "Digital control based on input-output data," *Proceedings of the 16th SICE Symposium on Control Theory*, pp. 47–50, 1987. (in Japanese)

[4] M. Ikeda, Y. Fujisaki, and N. Hayashi, "A model-less algorithm for tracking control based on input-output data," *Nonlinear Analysis*, Vol. 47, No. 3, pp. 1953–1960, 2001.

[5] U. S. Park and M. Ikeda, "Data-based stability analysis for linear time-invariant discrete-time systems," *Transactions of the Institute of Systems, Control and Information Engineers*, in press. (in Japanese)

[6] U. S. Park and M. Ikeda, "Data-based design of stabilizing controller for linear time-invariant discrete-time systems," *Transactions of the Institute of Systems, Control and Information Engineers*, unpublished. (in Japanese)

[7] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, London, 1998.