GLO AL AS MPTOTIC OUTPUT TRACKING FOR A CLASS OF NONLINEAR S STEMS

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Abstract: This paper considers a global asymptotic output tracking problem with a prescribed constant reference signal for a class of single-input and single output-output nonlinear systems. It is shown that under some mild conditions on such a system there is a smooth output feedback achieving global asymptotic output tracking and such a smooth output controller is explicitly constructed by a new design method proposed. The usefulness of our result is illustrated by a numerical example.

Keywords: nonlinear system, output tracking, output feedback, reference signal

1. INTRODUCTION

One of most important problems in control theory is to design a feedback control law making the output of a system asymptotically track a prescribed smooth reference signal. This problem has a long-standing history and has been thoroughly investigated over the last three decades.

The asymptotic tracking problem requires not only to asymptotically track a desired reference signal but also to guarantee the boundedness of all the internal states in the system. In general, even when the state of the system is known, tracking problems are much more difficult than those of stabilization. Thus solving asymptotic output tracking problems via output feedback is more difficult and challenging than solving those via state feedback (see [1]).

In this paper, we consider essentially the same class of nonlinear systems as treated in [2-4]. By far, it seems that one of most relaxed conditions imposed on the nonlinear terms of a given system is a triangular-type condition as far as the output feedback control is concerned as discussed in [2-3]. In [4], it has been proved that under a furthermore relaxed condition on the nonlinear term than a triangular-type condition, global asymptotic stabilization problems can be solved by output feedback.

In this paper, we deal essentially with the same class of nonlinear systems treated in [2-4], and study an output tracking problem with a prescribed constant reference signal for this class. It is shown that under some mild assumptions on a given system belonging to this class the global asymptotic output tracking problem is solved by smooth output feedback and further such a smooth output feedback is explicitly constructed. Finally the usefulness of our result is illustrated by a numerical example.

2. GLO AL AS MPTOTIC OUTPUT TRACKING FOR A CLASS OF NONLINEAR S STEMS OUTPUT FEED ACK

We consider a global asymptotic output tracking problem by output feedback for single-input and single output-output nonlinear systems of the form given by

$$\begin{aligned} \dot{x}_1 &= x_2 + \delta_1(t,x,u) \\ \dot{x}_2 &= x_3 + \delta_2(t,x,u) \\ &\vdots \\ \dot{x}_n &= u + \delta_n(t,x,u) \\ y &= x_1 \end{aligned} \tag{1}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in \boldsymbol{R}^n$ is the state, $u \in \boldsymbol{R}$ and $y \in \boldsymbol{R}$ are the input and the output of the system, respectively, the mapping $\delta_i : \boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}$ for $i = 1, \cdots, n$ are smooth with $\delta_i(0,\ldots,0) = 0$. The class of systems under consideration is assumed to satisfy the following condition.

Assumption 1. For System (1), there exists a function $\gamma(s) \geq 0$ such that for any $s > 0$ the inequality

$$\sum_{i=1}^{n} s^{i-1} \left|\delta_i(t,x,u)\right| \leq \gamma(s) \sum_{i=2}^{n} s^{i-1} \left|x_i\right| \tag{2}$$

is satisfied. □

System (1) can be rewritten in the following form:

$$\begin{aligned} \dot{x} &= Ax + Bu + \delta(t,x,u) \\ y &= Cx \end{aligned} \tag{3}$$

where

$$x = \left(x_1, \cdots, x_n\right)^T$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^T$$

$C = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$, $\delta(t,x,u) = \left(\delta_1(t,x,u), \cdots, \delta_n(t,x,u)\right)$.

Formally, the tracking problem can be formulated as follows: given a bounded reference signal $y_r(t)$ with bounded derivatives $y_r^{(1)}(t), \cdots, y_r^{(n)}(t)$, find, if possible, a dynamic output compensator of the form

$$\begin{aligned} \dot{\xi} &= \alpha(\xi, y, y_r, \dot{y}_r, \ldots, y_r^{(n)}), \qquad \xi \in R^n \\ u &= u(\xi, y, y_r, \dot{y}_r, \ldots, y_r^{(n)}) \end{aligned} \tag{4}$$

such that all the states of the closed system (1) with (4) are globally bounded; the output of this closed system with any initial state $\left(x(0), \xi(0)\right) \in R^n \times R^n$ satisfies

$$\lim_{t \to \infty} \left| y(t) - y_r(t) \right| = 0. \tag{5}$$

In the article [4], global asymptotic stabilization of system (1) can be solved by output feedback. In this paper, it will be shown that asymptotic output tracking is achievable by smooth output feedback. Further an output feedback controller solving this problem will be explicitly constructed by a new method. Indeed, the following result can be established.

Theorem 1. *Consider system (1) or equivalently (3) and suppose that it satisfies Assumption 1 and there exist matrices K, L and $s > 0$ such that the matrices*

$$A_K = A + BK, \quad A_L = A + LC$$

are stable and $\det W(s) > 0$ *where* $W(s)$ *defined as (16). Then the global asymptotical tracking problem for any constant reference signal $y_r(t) \equiv c$ is solvable by a smooth dynamic output feedback controller of the form (4).*

Proof. Let $y_r(t) \equiv c$ be a constant reference signal that needs to be tracked. Define the error signals by

$$e(t) = y(t) - y_r(t) = x_1 - c.$$

Then system (1) is equivalent to

$$\begin{aligned} \dot{e} &= x_2 + \delta_1(e + c, x_2, \ldots, x_n) \\ \dot{x}_2 &= x_3 + \delta_2(e + c, x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= u + \delta_n(e + c, x_2, \ldots, x_n) \\ y &= x_1, \end{aligned} \tag{6}$$

and further (2) in Assumption 1 is written as

$$\sum_{i=1}^{n} s^{i-1} \left| \delta_i(e + c, x_2, \ldots, x_n) \right| \le \gamma(s) \sum_{i=2}^{n} s^{i-1} \left| x_i \right| \tag{7}$$

Thus it becomes clear that solving the problem of global asymptotic tracking for system (1) is equivalent to achieving global stabilization of the error dynamic system (6).

Now, we design a linear output feedback controller

$$\begin{aligned} \dot{e} &= x_2 - \frac{L_1}{s}(e - e) \\ \dot{x}_2 &= x_3 - \frac{L_2}{s^2}(e - e) \\ &\vdots \\ \dot{x}_n &= u - \frac{L_n}{s^n}(e - e) \\ u &= \frac{K_1}{s^n} e + \frac{K_2}{s^{n-1}} x_2 + \cdots + \frac{K_n}{s} x_n \end{aligned} \tag{8}$$

or equivalent to

$$\dot{z} = Az + Bu - L(s)(y - Cz) \tag{9}$$

$$u = K(s) z \tag{10}$$

where $z = \left(e, x_2, \ldots, x_n\right)^T$,

$$K(s) = \left(\frac{K_1}{s^n}, \frac{K_2}{s^{n-1}}, \ldots, \frac{K_n}{s} \right)$$

and

$$L(s) = \left(\frac{L_1}{s}, \frac{L_2}{s^2}, \ldots, \frac{L_n}{s^n} \right)^T$$

with any $s > 0$.

First, we select $K = \left[K_1, \ldots, K_n\right]$ and $L = \left[L_1, \ldots, L_n\right]$ so as to satisfy that $A_K = A + BK$ and $A_L = A + LC$ are stable. Next defining

$$\varepsilon_1 = e - e, \;\; \varepsilon_i = x_i - x_i, \; 2 \le i \le n,$$

it follows from (6) and (8) that

$$\dot{\varepsilon} = A_L(s)\varepsilon + \delta(t, x, u) \tag{11}$$

where

$$A_L(s) = A + L(s)C.$$

Now, from (6) and (10), the closed loop system (6)-(9)-(10) becomes

$$\dot{z} = A_K(s) z + \delta(t, z, u) - BK(s)\varepsilon \tag{12}$$

where $z = \left(e, \, x_2, \, \cdots, \, x_n\right)^T$ and

$$A_K(s) = A + BK(s).$$

Further defining

$$D(s) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & s & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & s^{n-1} \end{pmatrix},$$

and noticing that A_K and A_L are stable, we have the following equalities:

$$sA_K(s) = D(s)^{-1} A_K D(s),$$
$$A_K^T(s)P_K(s) + P_K(s)A_K(s) = -s^{-1}D(s)^2$$
$$sA_L(s) = D(s)^{-1} A_L D(s)$$
$$A_L^T(s)P_L(s) + P_L(s)A_L(s) = -s^{-1}D(s)^2$$

where

$$P_K(s) = D(s)P_K D(s), \quad P_L(s) = D(s)P_L D(s),$$
$$A_K^T P_K + P_K A_K = -I, \quad A_L^T P_L + P_L A_L = -I.$$

Now, we set $V_1(\varepsilon) = \varepsilon^T P_L(s)\varepsilon$. Then, from (12), Assumption 1 and the previous equalities, we have

$$\begin{aligned}
\dot{V}_1(\varepsilon) &= -s^{-1}\|D(s)\varepsilon\|^2 + 2\varepsilon^T D(s)P_L D(s)\delta(t,z,u) \\
&\le -s^{-1}\|D(s)\varepsilon\|^2 + 2\|P_L\|\|D(s)\varepsilon\|\|D(s)\delta(t,x,u)\|_1 \\
&\le -s^{-1}\|D(s)\varepsilon\|^2 + 2\|P_L\|\|D(s)\varepsilon\|\sum_{i=1}^{n} s^{i-1}|\delta_i(t,z,u)| \\
&\le -s^{-1}\|D(s)\varepsilon\|^2 + 2\|P_L\|\|D(s)\varepsilon\|\gamma(s)\sum_{i=2}^{n} s^{i-1}|x_i| \\
&\le -s^{-1}\|D(s)\varepsilon\|^2 + 2\|P_L\|\|D(s)\varepsilon\|\gamma(s)\left(e + \sum_{i=2}^{n} s^{i-1}|x_i|\right) \\
&\le -s^{-1}\|D(s)\varepsilon\|^2 + 2\sqrt{n}\gamma(s)\|P_L\|\|D(s)\varepsilon\|\|D(s)z\|
\end{aligned} \tag{13}$$

Next, we set $V_2(z) = z^T P_K(s)z$ and note that

$$D(s)BK(s)\varepsilon = s^{-1}BKD(s)\varepsilon.$$

Then, from (12), Assumption1 and the previous equalities, we obtain

$$\begin{aligned}
\dot{V}_2(z) &\le -\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\}\|D(s)z\|^2 \\
&\quad - 2z^T D(s)P_K D(s)BK(s)\varepsilon \\
&= -\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\}\|D(s)z\|^2 \\
&\quad - 2s^{-1}z^T D(s)P_K BKD(s)\varepsilon \\
&\le -\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\}\|D(s)z\|^2 \\
&\quad + 2s^{-1}\|P_K\|\|K\|\|D(s)z\|\|D(s)\varepsilon\|
\end{aligned} \tag{14}$$

Now, we introduce a function

$$V = M_1V_1(\varepsilon) + M_2V_2(z), \; M_i > 0, \; i = 1, 2 \tag{15}$$

for the system (11) and (12). Then using (13) and (14), we have

$$\begin{aligned}
\dot{V} &\le -s^{-1}M_1\|D(s)\varepsilon\|^2 + 2M_1\sqrt{n}\gamma(s)\|P_L\|\|D(s)\varepsilon\|\|D(s)z\| \\
&\quad - M_2\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\}\|D(s)z\|^2 \\
&\quad + 2s^{-1}M_2\|P_K\|\|K\|\|D(s)z\|\|D(s)\varepsilon\| \\
&\le -s^{-1}M_1\|D(s)\varepsilon\|^2 \\
&\quad + 2\left(M_2s^{-1}\|P_K\|\|K\| + M_1\sqrt{n}\gamma(s)\|P_L\|\right)\|D(s)\varepsilon\|\|D(s)z\| \\
&\quad - M_2\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\}\|D(s)z\|^2 \\
&\le -\begin{bmatrix}\|D(s)\varepsilon\| \\ \|D(s)z\|\end{bmatrix}^T \cdot \begin{bmatrix} M_1s^{-1} & X(s) \\ X(s) & M_2\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\}\end{bmatrix} \cdot \begin{bmatrix}\|D(s)\varepsilon\| \\ \|D(s)z\|\end{bmatrix} \\
&= -\begin{bmatrix}\|D(s)\varepsilon\| \\ \|D(s)z\|\end{bmatrix}^T W(s) \begin{bmatrix}\|D(s)\varepsilon\| \\ \|D(s)z\|\end{bmatrix}
\end{aligned}$$

where

$$W(s) = \begin{bmatrix} M_1s^{-1} & X(s) \\ X(s) & M_2\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\}\end{bmatrix} \tag{16}$$

$$X(s) = -M_2s^{-1}\|P_K\|\|K\| - M_1\sqrt{n}\gamma(s)\|P_L\|.$$

By the condition

$$\det W(s) = M_1M_2s^{-1}\left\{s^{-1} - 2\sqrt{n}\|P_K\|\gamma(s)\right\} - X(s)^2 > 0$$

So, the closed-loop system (6)-(9)-(10) is globally exponentially stable. This implies the global output tracking problem with any constant signal $y_r(t) \equiv c$ is solvable, for the nonlinear system (1). □

Example. We work out a simple numerical example to illustrate the result described in Theorem 1. The example we consider is a 2-dimensional system of the following form:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \frac{x_1^2 x_2}{1 + 2x_1^2} \\
\dot{x}_2 &= u \\
y &= x_1
\end{aligned} \tag{17}$$

It is easy to verify that system (17) satisfies Assumption1. It follows from Theorem 1 that for any given constant reference signal $y_r(t) \equiv c$ there exists a dynamic output feedback controller of the form (8), achieving asymptotic output tracking for the nonlinear system (17). For instance, let

$$y_r(t) = 10.$$

Then, following the design procedure above, one can obtain the following output feedback controller

$$\begin{aligned}
\dot{e} &= x_2 + \frac{L_1}{s}(y - y_r - e) \\
\dot{x}_2 &= u + \frac{L_2}{s^2}(y - y_r - e) \\
u &= \frac{K_1}{s^2}e + \frac{K_2}{s}x_2
\end{aligned} \tag{18}$$

where

$K = [K_1, K_2] = [-2, -3]$ and $L = [L_1, L_2] = [-4, -4]^T$.

In view of Assumption1, we can check that $\gamma(s) = 1/2s$.

With $\gamma(s)=1/2s$, we choose $M_1=1$ and $M_2=s^2$. Then, there exists a range of s to satisfy $\det T(s)>0$. We choose $s=0.18$. The simulation results shown in Fig.1 or Fig.2 demonstrate asymptotic output tracking of (17) is achieved by the output compensator (18).

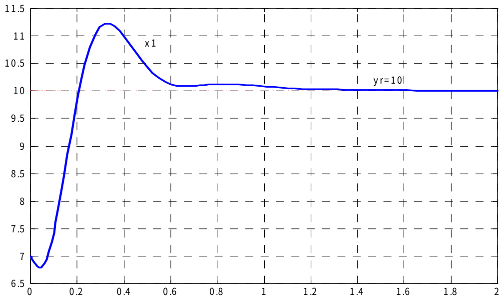

Fig.1(a). output reference

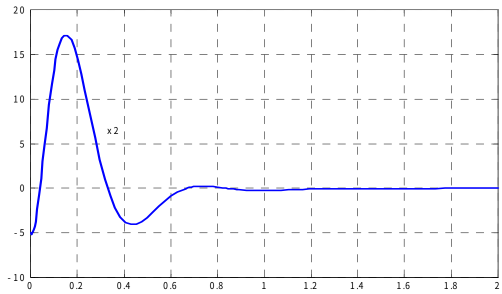

Fig.1(b). trajectory of state x_2 with $x_2(0)=-5$

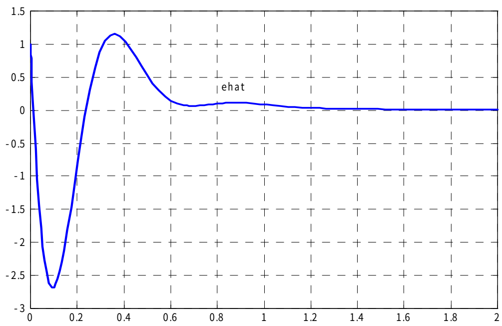

Fig.1(c). trajectory of e with $e(0)=1$

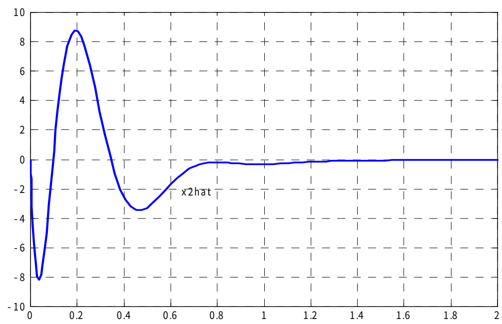

Fig.1(d). trajectory of state x_2 with $x_2(0)=0$

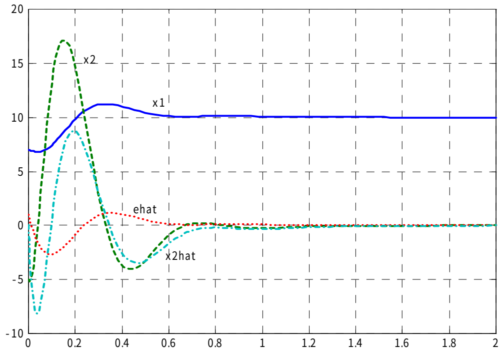

Fig.2. state trajectories with $x_1(0)=7, x_2(0)=-5, e(0)=1, x_2(0)=0$

3. CONCLUSIONS

In this paper, the asymptotic output tracking problem for a system belonging to a class of nonlinear systems was studied. It was shown that the asymptotic output tracking problem is solvable by a smooth output feedback controller and such a controller can be explicitly constructed. Finally a simple numerical example was work out to illustrate the result obtained.

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