Stability Condition of Discretized Equivalent Control Based Sliding mode Controller for Second-Order Systems with external disturbance

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Abstract: A novel sufficient condition of discretized equivalent control based sliding mode controller (SMC) for a second-order system with external disturbance to be globally uniformly ultimately bounded (GUUB) is proposed. The proposed stability condition guarantees that the system state is always GUUB in the presence of disturbance. The ultimate bounds of the system state variables are also derived.

Keywords: Stability condition, Discretized system, Sliding mode control, GUUB.

1. INTRODUCTION

Almost all of studies of sliding mode control (SMC) have been proposed in the continuous-time domain [1-2]. In the actual system, however, controllers are implemented in the discrete-time domain since they use microprocessors and/or digital computers. Recently, discrete-time sliding mode control (DSMC) has been studied extensively to address various controllers using specific principles [3-6]. However, the research of discretizing a continuous-time SMC for digital implementation has not been fully explored. Furthermore, it is also well known that a control system designed in the continuous-time domain may become unstable after sampling.

Recently chaotic behaviors were found in discretizing continuous SMC systems by X. Yu[7-8]. Yu and Chen proposed the the sufficient conditions for the discretized system to be GUUB[9] . But these studies can be only applied under the nonexistence of external disturbances.

In this paper, therefore, a novel sufficient condition of discretized equivalent control based sliding mode controller (SMC) for a second-order system with external disturbance to be GUUB is proposed. The proposed stability condition guarantees that the system state is always GUUB in the presence of disturbance. The ultimate bounds of the system state variables are also derived. Finally, simulation results are presented to show the effectiveness of the proposed method.

2. DISCRETIZATION OF AN EQUIVALENT CONTROL BASED SECOND-ORDER SMC

Consider a second-order linear plant of the following form

$$\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,, \qquad (1)$$

where $x \notin R^2$ is the state vector, $u \notin R^1$ is the system input, and a_1, a_2 are elements of a system matrix. Let the sliding surface be $\sigma = c^T x = \left[c_1\ 1\right] x$, where $c_1 > 0$ is assumed to be designed such that that sliding dynamics, $\sigma = 0$, are asymptotically stable. From $\dot{\sigma} = 0$, we can easily obtain the equivalent control law as

$$\dot{\sigma} = c^T \dot{x} = c^T Ax + c^T bu = 0 \quad \Rightarrow u_{eq} = -(c^T b)^{-1} c^T Ax.$$

From the sliding mode existence condition, $\sigma\dot{\sigma} < 0$, we have the following equivalent control based SMC:

$$\begin{aligned} u &= u_{eq} + u_s \\ &= -(c^T b)^{-1} c^T Ax - \alpha (c^T b)^{-1} \operatorname{sgn}(\sigma(x)), \end{aligned} \qquad (2)$$

where $\alpha > 0$ is a control gain, and $\operatorname{sgn}(\cdot)$ is a signum function. Clearly $c^T b$ is nonsingular.

To discretize the overall system, we convert the continuous-time system (1) under the zero-order hold (ZOH) to the discrete-time system

$$x(k+1) = e^{Ah} x(k) + \int_0^h e^{A\tau} d\tau b u(k), \qquad (3)$$

where

$$\begin{aligned} u(k) &= u_{eq}(k) + u_s(k) \\ &= -c^T Ax(k) - \alpha \cdot \operatorname{sgn}(\sigma(x(k))) \,, \end{aligned} \qquad (4)$$

h is a sampling period, and the index k indicates the $k-$th sample.

As the system state $x(k)$ evolves, the switching function $\operatorname{sgn}(\sigma(x(k)))$ forms a sequence of binary values of -1 and $+1$. For simplicity, we denote $\operatorname{sgn}(\sigma(x(k)))$ as $s_k \in \{-1, 1\}$. Then the discretized system can be described by

$$x(k+1)=\Phi x(k)+\alpha\Gamma s_k = \begin{bmatrix} 1 & v(h) \\ 0 & d(h) \end{bmatrix} x(k)+\alpha \begin{bmatrix} \gamma_1(h) \\ \gamma_2(h) \end{bmatrix} s_k, \quad (5)$$

where $\Phi = e^{Ah} - \int_0^h e^{A\tau} d\tau b c^T A$, $\Gamma = \int_0^h e^{A\tau} d\tau b$.

To calculate Φ and Γ, the matrix e^{Ah} is derived as

$$e^{Ah} = e^{-\beta h} \cdot \begin{bmatrix} \cos\zeta h + \beta\zeta^{-1}\sin\zeta h & \zeta^{-1}\sin\zeta h \\ -a_1\zeta^{-1}\sin\zeta h & \cos\zeta h - \beta\zeta^{-1}\sin\zeta h \end{bmatrix}, \quad (6)$$

where $\beta = a_2/2$, $\zeta = (1/2)\sqrt{4a_1 - a_2^2}$.

If $4a_1 - a_2^2 > 0$, then v, d, γ_1, and γ_2 can be derived as

$$v = \zeta^{-1} e^{-\beta h}\sin\zeta h - \frac{(c_1-a_2)\cdot(-e^{-\beta h}\zeta^{-1}(\beta\sin\zeta h+\zeta\cos\zeta h)+1)}{\beta^2+\zeta^2}, \quad (7)$$

$$d = e^{-\beta h}(\cos\zeta h + (\beta - c_1)\zeta^{-1}\sin\zeta h), \quad (8)$$

$$\gamma_1 = \frac{e^{-\beta h}\zeta^{-1}(-\beta\sin\zeta h - \zeta\cos\zeta h)+1}{\beta^2+\zeta^2}, \quad (9)$$

$$\gamma_2 = \zeta^{-1}e^{-\beta h}\sin\zeta h. \quad (10)$$

Using above equations, the discretized second-order dynamics with external disturbance can be rewritten as

$$x_1(k+1) = x_1(k) + v x_2(k) - \gamma_1\alpha s_k + w_d(k), \quad (11)$$

$$x_2(k+1) = d x_2(k) - \gamma_2\alpha s_k + w_d(k), \quad (12)$$

where w_d is the external disturbance with mean value m_d and it is assumed that there exists a constant vector $\xi \in \Re^n$ such that $|w_d(k) - m_d| < \xi \ \forall k$, where $k = 1,2,\cdots,n$.

3. STABILITY CONDITION OF DISCRETIZED SMC

Generally, the asymptotic stability can be guaranteed if the sliding mode controller with a constant control gain is implemented in the continuous-time domain. For the discrete-time system, however, the ultimate boundedness can be ensured. In the following theorem, we derive conditions for the stability of the closed-loop system with the discretized equivalent control based SMC (4).

Theorem 1: For the discretized systems (11) ~ (12) with the discretized equivalent control based SMC (4), the overall system is globally uniformly ultimately bounded (GUUB) if

$$|d| < 1, \quad (13)$$

and

$$\alpha > \left| \frac{(1-d+v)}{-v\gamma_2 - \gamma_1(1-d)} \right| (|m_d| + \xi). \quad (14)$$

Furthermore, the ultimate bounds of the system state variables are given by

$$\lim_{k\to\infty} |x_1(\infty)| \le \left(\gamma_1\alpha + \frac{(c_1^{-1}-v)(|\gamma_2|\alpha + |m_d| + \xi)}{1-|d|} \right) + (|m_d| + \xi), \quad (15)$$

$$\lim_{k\to\infty} |x_2(\infty)| \le \frac{|\gamma_2|\alpha + |m_d| + \xi}{1-|d|}. \quad (16)$$

Proof: From (12), if we let $\xi = 0 \ \forall k$, it is clear that (13) has to be satisfied because the pole of the system (12) should be located inside the unit circle. It is also obvious that the ultimate bound of x_2 is on the equilibrium line, (11) can be rewritten as

$$x_1(k+1) = x_1(k) + v\left(\frac{-\gamma_2\alpha s_k + m_d}{1-d}\right) - \gamma_1\alpha s_k + m_d. \quad (17)$$

In order to the state x_1 converges to the sliding surface, the last three terms of (17) should satisfy the following inequality:

$$v\left(\frac{-\gamma_2\alpha s_k + m_d}{1-d}\right) - \gamma_1\alpha s_k + m_d < 0. \quad (18)$$

Since $(1-d) > 0$ and $|s_k| = 1$, (18) can be written finally as

$$\alpha > G_\alpha = \left| \frac{(1-d+v)}{-v\gamma_2 - \gamma_1(1-d)} \right| |m_d|, \quad (19)$$

where G_α is rate for input s_k and disturbance m_d.

Therefore, if we select α greater than G_α, discretized system can be controlled, whereas if we select α less than G_α discretized system becomes unstable.

For the case of $\xi \ne 0 \ \forall k$, phase portrait has the chattering phenomenon around equilibrium line. And since the value of G_α is in proportion to the magnitude of disturbance, G_α is obtained by $|(1-d+v)/(-v\gamma_2 - \gamma_1(1-d))|(|m_d| + \xi)$.

The ultimate bound of the state x_1 can be derived by considering the switching points – intersection of the sliding surface and the equilibrium lines as can be seen in Figure 1. Since the points are on the sliding surface and the equilibrium line, x_2 should have a value of its ultimate bound, and x_1 has to satisfy

$$x_1(k) = -c_1^{-1} x_2(k) \ . \tag{20}$$

Substituting (20) into (11) gives

$$x_1(k+1) = \frac{(c_1^{-1} - \nu)(|\gamma_2|\alpha + |m_d| + \xi)}{1 - |d|} - \gamma_1 \alpha s_k + (|m_d| + \xi). \tag{21}$$

From (21), therefore, the ultimate bound of x_1 can be obtained as (15). ■

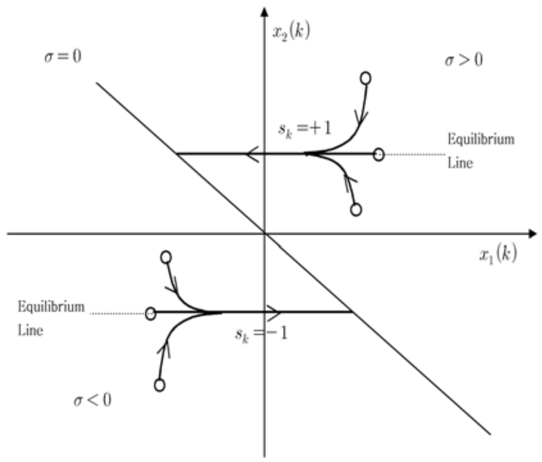

Fig. 1 Phase portrait for several initial conditions

4. SIMULATION STUDIES

Consider the following continuous-time system as, $a_1 = 100$, $a_2 = 0$, $c_1 = 1$. Let $h = 0.1$ and $x(0) = (0.5,\ 0.5)$. And we choose disturbance with $m_d = 0.01$ and $\xi = 0$. First, by Theorem 1, G_α is calculated as 0.68, From Figure 2, if the value of α equals G_α, then it is seen that the trajectory converges to one fixed point which are $(0.59,\ -0.09)$. If the value of α is bigger than G_α, then system state approach some specified boundaries of system steady states as Figure 3, and it shows that system is stable for all values of $\alpha > G_\alpha$. However, system is unstable for all values of $\alpha < G_\alpha$.

We now look at another interesting phenomenon. Let $h = 0.5$ and $x(0) = (6.0,\ 6.0)$. And we choose disturbance with $m_d = 0.5$ and $\xi = 0.17$. First, by Theorem 1, the G_α is 24.3. For the first case of $\alpha = 25$, the theoretical values of the boundary is $|x_1| < 5.38$, $|x_2| < 4.94$. And as can be seen in Figure 5, the states are uniformly bounded by estimated boundary. For the second case of $\alpha = 17$, discretized system becomes unstable as Figure 6.

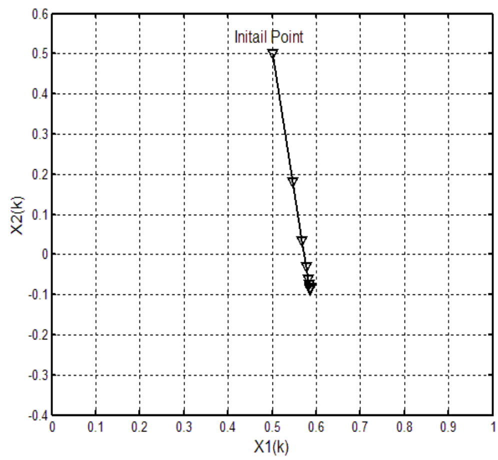

Fig. 2 Phase portrait for $\alpha = G_\alpha$

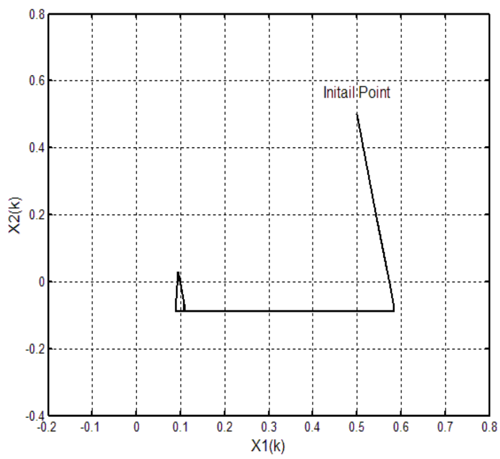

Fig. 3 Phase portrait for $\alpha > G_\alpha$

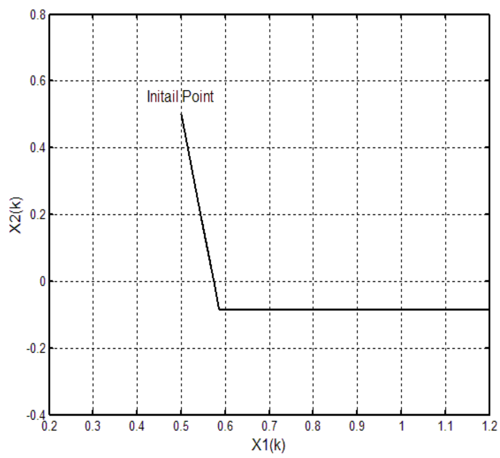

Fig. 4 Phase portrait for $\alpha < G_\alpha$

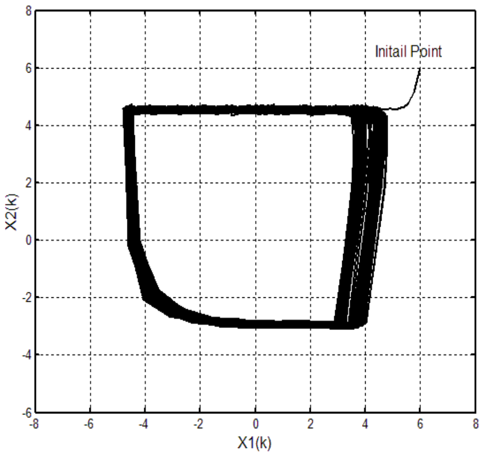

Fig. 5 Phase portrait for $\alpha = 25$

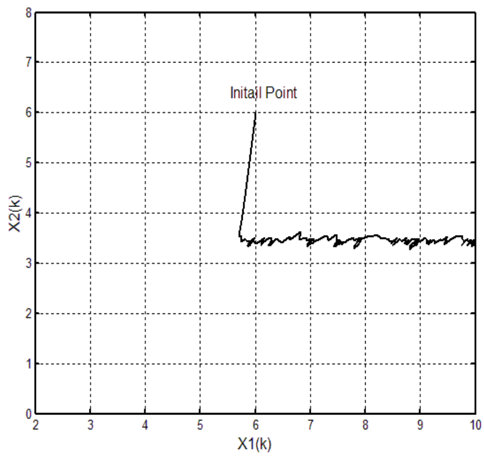

Fig. 6 Phase portrait for $\alpha = 17$

5. CONCLUSION

In this paper, a condition of discretized equivalent control based sliding mode controller (SMC) for a second-order system with external disturbance to be GUUB has been presented. It has been shown that the system state is GUUB in the presence of disturbance.

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