Extension and Simplification of Inverse LQ Regulator of Large Scale Systems by Decentralized Control

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Abstract: An LMI based method to construct a decentralized control law for large scale systems is discussed. It is extended to assure the stability not only of the overall system but also of each subsystem without interconnection. Then, it is simplified to have local feedback loops only for some selected subsystems.

Keywords: large scale system, decentralized control, linear quadratic regulator, linear matrix inequality

1. Introduction

Decentralized control is known to be a practical way to stabilize a large scale system which consists of multiple subsystems. Geromel and Bernussou proposed a method to obtain the optimal decentralized control via the mathematical programming[1], [2]. Ikeda and Šiljak constructed a linear quadratic (LQ) regulator of each subsystem, and modified the parameters to make the overall closed loop system an LQ regulator[3], [4].

Recently, the present author proposed a method to construct a decentralized control law for large scale systems[5]. The feedback gain was calculated with a solution of linear matrix inequalities (LMIs). The resulting closed loop system belonged to a class of LQ regulators, so it was assured to have the good robustness properties [6], [7]. The method proposed in [5] assured only the stability of the overall closed loop system.

In this paper, firstly, an extended method, which assures the stability not only of the overall system but also of each subsystem without interconnection, of the method presented in [5] is proposed. The LMIs with some extra inequalities corresponding to the stability of each subsystem are solved to obtain the decentralized feedback gain. Next, the simplification of the control structure is considered. In the method in [5], all subsystems were treated equally, but in practice, it is not necessary to control all subsystems. So, a method to reduce the number of feedback loops is proposed. In this method, only some selected subsystems are controlled. Subsystems not to be controlled are assumed to have no input channels, that is, the input matrices are forced to be zero matrices, and decentralized feedback gain is calculated. If the decentralized control stabilizing the overall closed loop system does exist, then it means that it is not necessary to control the subsystems of zero input matrices. So subsystems are divided into two groups: one group to be controlled and the other group not to be controlled. There may be various combinations of controlled and not controlled groups. One way to find the optimal set among them is to introduce another cost function, which is the summation of the costs defined to each decentralized controller. For some of the selections, the decentralized controller may not be found. For the other of the selections, it may be found. Among the latter selections, we should choose the set of controlled subsystems to minimize the newly introduced cost function. Even some of the subsystems are not controlled, the resulting overall closed loop system is still stable and belongs to a class of LQ regulators, so it is assured to have good robustness properties.

2. Preliminary Results

Let us consider a large scale system consists of N interconnected subsystems described by

$$\dot{x}_i(t) = \sum_{j=1}^{N} A_{ij} x_j(t) + B_i u_i(t) \quad (i = 1, 2, \cdots, N) \tag{1}$$

$$x_i(0) = x_{i0} \quad (i = 1, 2, \cdots, N)$$

where $x_i(t) \in \mathbf{R}^{n_i}$ is the state of the i-th subsystem, $u_i(t) \in \mathbf{R}^{m_i}$ is the input of the i-th subsystem, and $A_{ij} \in \mathbf{R}^{n_i \times n_j}$, $B_i \in \mathbf{R}^{n_i \times m_i}$ are constant matrices. $A_{ij} (i \neq j)$ denotes the interconnection from the j-th subsystem to the i-th subsystem. x_{i0} is the initial state of the i-th subsystem. The overall system consists of subsystems (1) can be expressed as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

$$x(0) = x_0$$

where

$$x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) & \cdots & x_N^T(t) \end{bmatrix}^T,$$

$$u(t) = \begin{bmatrix} u_1^T(t) & u_2^T(t) & \cdots & u_N^T(t) \end{bmatrix}^T,$$

$$x_0 = \begin{bmatrix} x_{10}^T & x_{20}^T & \cdots & x_{N0}^T \end{bmatrix}^T,$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$B = \mathrm{blockdiag}\begin{pmatrix} B_1, & B_2, & \cdots, & B_N \end{pmatrix}.$$

The plant parameters are assumed to satisfy the following condition.

Condition 1: There exist symmetric matrices $S_i \in \mathbf{R}^{n_i \times n_i}$ $(i = 1, 2, \cdots, N)$ and $T_i \in \mathbf{R}^{m_i \times m_i}$ $(i =$

$1, 2, \cdots, N$) which satisfy the following LMIs.

$$L = \begin{bmatrix} -S_1A_{11}^T - A_{11}S_1 + B_1T_1B_1^T & -A_{12}S_2 - S_1A_{21}^T & \cdots & -A_{1N}S_N - S_1A_{N1}^T \\ -S_2A_{12}^T - A_{21}S_1 & -S_2A_{22}^T - A_{22}S_2 + B_2T_2B_2^T & \cdots & -A_{2N}S_N - S_2A_{N2}^T \\ \vdots & \vdots & \ddots & \vdots \\ -S_NA_{1N}^T - A_{N1}S_1 & -S_NA_{2N}^T - A_{N2}S_2 & \cdots & -S_NA_{NN}^T - A_{NN}S_N + B_NT_NB_N^T \end{bmatrix} > 0 \tag{3}$$

$$S_i > 0 \quad (i = 1, 2, \cdots, N) \tag{4}$$

$$T_i > 0 \quad (i = 1, 2, \cdots, N) \tag{5}$$

□

With these S_i's and T_i's,

$$P_i = S_i^{-1} > 0 \quad (i = 1, 2, \cdots, N) \tag{6}$$

$$R_i = T_i^{-1} > 0 \quad (i = 1, 2, \cdots, N) \tag{7}$$

are determined and a decentralized feedback law for each subsystem (1) is constructed as

$$u_i(t) = -R_i^{-1}B_i^TP_ix_i(t) \quad (i = 1, 2, \cdots, N) \tag{8}$$

or

$$u(t) = -R^{-1}B^TPx(t) \tag{9}$$

to give the closed loop system

$$\dot{x}_i(t) = \sum_{j=1}^{N} A_{ij}x_j(t) - B_iR_i^{-1}B_i^TP_ix_i(t) \tag{10}$$
$$(i = 1, 2, \cdots, N)$$

or

$$\dot{x}(t) = (A - BR^{-1}B^TP)x(t) \tag{11}$$

is obtained where

$$P = \text{blockdiag}\begin{pmatrix} P_1, & P_2, & \cdots, & P_N \end{pmatrix} > 0$$

$$R = \text{blockdiag}\begin{pmatrix} R_1, & R_2, & \cdots, & R_N \end{pmatrix} > 0.$$

If we define

$$Q = \begin{bmatrix} -A_{11}^TP_1 - P_1A_{11} + P_1B_1R_1^{-1}B_1^TP_1 & -P_1A_{12} - A_{21}^TP_2 & \cdots & -P_1A_{1N} - A_{N1}^TP_N \\ -A_{12}^TP_1 - P_2A_{21} & -A_{22}^TP_2 - P_2A_{22} + P_2B_2R_2^{-1}B_2^TP_2 & \cdots & -P_2A_{2N} - A_{N2}^TP_N \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1N}^TP_1 - P_NA_{N1} & -A_{2N}^TP_2 - P_NA_{N2} & \cdots & -A_{NN}^TP_N - P_NA_{NN} + P_NB_NR_N^{-1}B_N^TP_N \end{bmatrix} \tag{12}$$

the following lemma was shown[5].

Lemma 1: Under Condition 1, the resulting overall closed loop system (11) is asymptotically stable, and it is the LQ regulator minimizing the cost functional

$$J = \int_0^\infty \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt. \tag{13}$$

□

In its consequence, it is assured to have insensitivity property and good stability margin [6], [7].

3. Extension

The method in [5] assured only the stability of the overall closed loop system. In this section, an extended method, which assures the stability not only of the overall system but also of each subsystem without interconnection, is proposed.

3.1. Cutting All Interconnections

The plant parameters are assumed to satisfy the following condition.

Condition 2: There exist symmetric matrices $S_i \in \mathbf{R}^{n_i \times n_i}$ $(i = 1, 2, \cdots, N)$ and $T_i \in \mathbf{R}^{m_i \times m_i}$ $(i = 1, 2, \cdots, N)$ which satisfy the following LMIs.

$$L = \begin{bmatrix} -S_1A_{11}^T - A_{11}S_1 + B_1T_1B_1^T & -A_{12}S_2 - S_1A_{21}^T & \cdots & -A_{1N}S_N - S_1A_{N1}^T \\ -S_2A_{12}^T - A_{21}S_1 & -S_2A_{22}^T - A_{22}S_2 + B_2T_2B_2^T & \cdots & -A_{2N}S_N - S_2A_{N2}^T \\ \vdots & \vdots & \ddots & \vdots \\ -S_NA_{1N}^T - A_{N1}S_1 & -S_NA_{2N}^T - A_{N2}S_2 & \cdots & -S_NA_{NN}^T - A_{NN}S_N + B_NT_NB_N^T \end{bmatrix} > 0 \tag{14}$$

$$-S_iA_{ii}^T - A_{ii}S_i + B_iT_iB_i^T > 0 \quad (i = 1, 2, \cdots, N) \tag{15}$$

$$S_i > 0 \quad (i = 1, 2, \cdots, N) \tag{16}$$

$$T_i > 0 \quad (i = 1, 2, \cdots, N) \tag{17}$$

□

The decentralized feedback law is constructed as (8) and the closed loop system is formed as (10). Then the following theorem holds.

Theorem 1: Under Condition 2, the resulting overall closed loop system (11) is asymptotically stable, and it is the LQ regulator minimizing the cost functional

$$J = \int_0^\infty \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt. \tag{18}$$

Moreover, even if all the interconnections are cut, each closed loop system is still stable and it is the LQ regulator minimizing the cost functional

$$J_i = \int_0^\infty \left\{ x_i^T(t) Q_i x_i(t) + u_i^T(t) R_i u_i(t) \right\} dt \tag{19}$$

where

$$Q_i = -A_{ii}^T P_i - P_i A_{ii} + P_i B_i R_i^{-1} B_i^T P_i. \quad (i = 1, 2, \cdots, N) \tag{20}$$

(proof) When all the interconnections are alive, Lemma 1 assures the first half of the theorem. Let us consider the case that all the interconnections are cut. Pre- and post multiplying P_i to (15) gives $Q_i > 0$. This means that the isolated subsystems

$$\dot{x}_i(t) = A_{ii} x_i(t) + B_i u_i(t) \quad (i = 1, 2, \cdots, N) \tag{21}$$

with the local feedback (8) is an LQ regulator minimizing the cost function (19), so the closed loop subsystems are all stable.

□

3.2. Cutting Some Interconnections

Next, let us consider the case that some of the interconnections may be cut but others not. If the interconnection from the j-th subsystem to the i-th subsystem may be cut, let

$$\bar{A}_{ij} = 0 \tag{22}$$

and if it may not be cut, let

$$\bar{A}_{ij} = A_{ij}. \tag{23}$$

The plant parameters are assumed to satisfy the following condition.

Condition 3: There exist symmetric matrices $S_i \in \mathbf{R}^{n_i \times n_i}$ $(i = 1, 2, \cdots, N)$ and $T_i \in \mathbf{R}^{m_i \times m_i}$ $(i = 1, 2, \cdots, N)$ which satisfy the following LMIs.

$$L = \begin{bmatrix} -S_1 A_{11}^T - A_{11} S_1 + B_1 T_1 B_1^T & -A_{12} S_2 - S_1 A_{21}^T & \cdots & -A_{1N} S_N - S_1 A_{N1}^T \\ -S_2 A_{12}^T - A_{21} S_1 & -S_2 A_{22}^T - A_{22} S_2 + B_2 T_2 B_2^T & \cdots & -A_{2N} S_N - S_2 A_{N2}^T \\ \vdots & \vdots & \ddots & \vdots \\ -S_N A_{1N}^T - A_{N1} S_1 & -S_N A_{2N}^T - A_{N2} S_2 & \cdots & -S_N A_{NN}^T - A_{NN} S_N + B_N T_N B_N^T \end{bmatrix} > 0 \tag{24}$$

$$\bar{L} = \begin{bmatrix} -S_1 A_{11}^T - A_{11} S_1 + B_1 T_1 B_1^T & -\bar{A}_{12} S_2 - S_1 \bar{A}_{21}^T & \cdots & -\bar{A}_{1N} S_N - S_1 \bar{A}_{N1}^T \\ -S_2 \bar{A}_{12}^T - \bar{A}_{21} S_1 & -S_2 A_{22}^T - A_{22} S_2 + B_2 T_2 B_2^T & \cdots & -\bar{A}_{2N} S_N - S_2 \bar{A}_{N2}^T \\ \vdots & \vdots & \ddots & \vdots \\ -S_N \bar{A}_{1N}^T - \bar{A}_{N1} S_1 & -S_N \bar{A}_{2N}^T - \bar{A}_{N2} S_2 & \cdots & -S_N A_{NN}^T - A_{NN} S_N + B_N T_N B_N^T \end{bmatrix} > 0 \tag{25}$$

$$S_i > 0 \quad (i = 1, 2, \cdots, N) \tag{26}$$

$$T_i > 0 \quad (i = 1, 2, \cdots, N) \tag{27}$$

□

The decentralized feedback law is constructed as (8) and the closed loop system is formed as (10). Then the following theorem holds.

Theorem 2: Under Condition 3, the resulting overall closed loop system (11) is asymptotically stable, and it is the LQ regulator minimizing the cost functional

$$J = \int_0^\infty \left\{ x^T(t) Q x(t) + u^T(t) R u(t) \right\} dt. \tag{28}$$

Moreover, even if the previously assigned interconnections are cut, the closed loop system is still stable and it is the LQ regulator minimizing the cost functional

$$\bar{J} = \int_0^\infty \left\{ x^T(t) \bar{Q} x(t) + u^T(t) R u(t) \right\} dt \tag{29}$$

where

$$\bar{Q} = \begin{bmatrix} -A_{11}^T P_1 - P_1 A_{11} + P_1 B_1 R_1^{-1} B_1^T P_1 & -P_1 \bar{A}_{12} - \bar{A}_{21}^T P_2 & \cdots & -P_1 \bar{A}_{1N} - \bar{A}_{N1}^T P_N \\ -\bar{A}_{12}^T P_1 - P_2 \bar{A}_{21} & -A_{22}^T P_2 - P_2 A_{22} + P_2 B_2 R_2^{-1} B_2^T P_2 & \cdots & -P_2 \bar{A}_{2N} - \bar{A}_{N2}^T P_N \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{A}_{1N}^T P_1 - P_N \bar{A}_{N1} & -\bar{A}_{2N}^T P_2 - P_N \bar{A}_{N2} & \cdots & -A_{NN}^T P_N - P_N A_{NN} + P_N B_N R_N^{-1} B_N^T P_N \end{bmatrix} \tag{30}$$

(proof) When all the interconnections are alive, Lemma 1 assures the first half of the theorem. Let us consider the case that the previously assigned interconnections are cut. Pre- and post multiplying P to (25) gives

$$\bar{Q} = -P\bar{A} - \bar{A}^T P + PBR^{-1}B^T P > 0 \tag{31}$$

where

$$\bar{A} = \begin{bmatrix} A_{11} & \bar{A}_{12} & \cdots & \bar{A}_{1N} \\ \bar{A}_{21} & A_{22} & \cdots & \bar{A}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{N1} & \bar{A}_{N2} & \cdots & A_{NN} \end{bmatrix}.$$

This means that the overall system

$$\dot{x}(t) = (\bar{A} - BR^{-1}B^T P)x(t) \tag{32}$$

is an LQ regulator minimizing the cost functional (29), so the closed loop systems is stable.

□

Remark: Theorem 2 shows the stability only of two cases: one case that all the interconnections are alive, and the other case that all the previously assigned interconnections are cut. If there are more cases for which the stability should be retained, the LMI condition can be added in a same manner. It is a feature of the LMI based approaches.

4. Simplification

In the above method, all subsystems must be controlled, but in practice, it is not necessary to control all subsystems. So, in this section, a method to reduce the number of feedback loops is proposed, where only some selected subsystems are controlled.

4.1. Feedback Loop Reduction

Let us divide subsystems into two sets: subsystems to be controlled (S_c) and others not to be controlled (S_r). If the i-th subsystem belongs to S_r, let

$$\tilde{B}_i = 0 \tag{33}$$

otherwise, let

$$\tilde{B}_i = B_i. \tag{34}$$

This means that the subsystems not to be controlled are regarded as ones without the input channel. The plant parameters are assumed to satisfy the following condition.

Condition 4: There exist symmetric matrices $S_i \in \mathbf{R}^{n_i \times n_i}$ $(i = 1, 2, \cdots, N)$ and $T_i \in \mathbf{R}^{m_i \times m_i}$ $(i = 1, 2, \cdots, N)$ which satisfy the following LMIs.

$$\tilde{L} = \begin{bmatrix} -S_1 A_{11}^T - A_{11} S_1 + \tilde{B}_1 T_1 \tilde{B}_1^T & -A_{12} S_2 - S_1 A_{21}^T & \cdots & -A_{1N} S_N - S_1 A_{N1}^T \\ -S_2 A_{12}^T - A_{21} S_1 & -S_2 A_{22}^T - A_{22} S_2 + \tilde{B}_2 T_2 \tilde{B}_2^T & \cdots & -A_{2N} S_N - S_2 A_{N2}^T \\ \vdots & \vdots & \ddots & \vdots \\ -S_N A_{1N}^T - A_{N1} S_1 & -S_N A_{2N}^T - A_{N2} S_2 & \cdots & -S_N A_{NN}^T - A_{NN} S_N + \tilde{B}_N T_N \tilde{B}_N^T \end{bmatrix} > 0 \tag{35}$$

$$S_i > 0 \quad (i = 1, 2, \cdots, N) \tag{36}$$

$$T_i > 0 \quad (i = 1, 2, \cdots, N) \tag{37}$$

□

Using these S_i's and T_i's,

$$P_i = S_i^{-1} > 0 \quad (i = 1, 2, \cdots, N) \tag{38}$$

$$R_i = T_i^{-1} > 0 \quad (i = 1, 2, \cdots, N) \tag{39}$$

are calculated and a decentralized feedback law for each subsystem (1) is constructed as

$$u_i(t) = -R_i^{-1} \tilde{B}_i^T P_i x_i(t) \quad (i = 1, 2, \cdots, N) \tag{40}$$

or

$$u(t) = -R^{-1} \tilde{B}^T P x(t). \tag{41}$$

where

$$\tilde{B} = \text{blockdiag} \begin{pmatrix} \tilde{B}_1 & \tilde{B}_2 & \cdots & \tilde{B}_N \end{pmatrix}.$$

Then the closed loop system

$$\dot{x}_i(t) = \sum_{j=1}^{N} A_{ij} x_j(t) - \tilde{B}_i R_i^{-1} \tilde{B}_i^T P_i x_i(t) \tag{42}$$

$$(i = 1, 2, \cdots, N)$$

or

$$\dot{x}(t) = (A - \tilde{B} R^{-1} \tilde{B}^T P)x(t) \tag{43}$$

is obtained where

$$P = \text{blockdiag} \begin{pmatrix} P_1, & P_2, & \cdots, & P_N \end{pmatrix} > 0$$

$$R = \text{blockdiag} \begin{pmatrix} R_1, & R_2, & \cdots, & R_N \end{pmatrix} > 0.$$

Theorem 3: Under Condition 4, the resulting overall closed loop system (43) is stable and it is the LQ regulator minimizing the cost functional

$$\tilde{J} = \int_0^\infty \left\{ x^T(t) \tilde{Q} x(t) + u^T(t) R u(t) \right\} dt \tag{44}$$

where

$$\tilde{Q} = \begin{bmatrix} -A_{11}^T P_1 - P_1 A_{11} + P_1 \tilde{B}_1 R_1^{-1} \tilde{B}_1^T P_1 & -P_1 A_{12} - A_{21}^T P_2 & \cdots & -P_1 A_{1N} - A_{N1}^T P_N \\ -A_{12}^T P_1 - P_2 A_{21} & -A_{22}^T P_2 - P_2 A_{22} + P_2 \tilde{B}_2 R_2^{-1} \tilde{B}_2^T P_2 & \cdots & -P_2 A_{2N} - A_{N2}^T P_N \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1N}^T P_1 - P_N A_{N1} & -A_{2N}^T P_2 - P_N A_{N2} & \cdots & -A_{NN}^T P_N - P_N A_{NN} + P_N \tilde{B}_N R_N^{-1} \tilde{B}_N^T P_N \end{bmatrix} \tag{45}$$

(proof) Pre- and post multiplying P to (35) gives

$$\tilde{Q} = -PA - A^T P + P \tilde{B} R^{-1} \tilde{B}^T P > 0. \tag{46}$$

This means that the overall closed loop system is an LQ regulator minimizing the cost functional (44), so the closed loop system is stable.

□

4.2. Grouping of Controlled Subsystems

If the i-th subsystem belongs to S_r,

$$\tilde{B}_i = 0 \tag{47}$$

means

$$u_i(t) = 0, \tag{48}$$

that is, the i-th subsystem is not needed to be controlled.
There may be various combinations of controlled and not controlled groups. One way to find the optimal set among them is to introduce another cost functional

$$C = \sum_i c_i \tag{49}$$

where c_i is the cost of the decentralized controller for the i-th subsystem.
For some of the selections, the decentralized controller may not be found. For the other of the selections, it may be found. Among the latter selections, we should choose the set of controlled subsystems to minimize the newly introduced cost function. Even if some of the subsystems are not controlled, the resulting overall closed loop system is still stable and belongs to a class of LQ regulators, so it is assured to have good robustness properties.

5. Conclusion

An LMI based method to construct an optimal decentralized feedback law for large scale systems was discussed. It was extended to assure the stability not only of the overall system but also of the subsystems without interconnections. It was also simplified so that not all but only some of the subsystems were needed to be controlled.

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