To Determine the Characteristic Polynomial Coefficients Based On the Transient Response

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Abstract: This paper presents a method to determine the characteristic polynomial of a closed loop all-pole system in order to obtain desired transient response in terms of the overshoot and speed (rising/settling time). The method adjusts the overshoot by doing some changes in the characteristic ratios of the Bessel-Thompson filter. The closed loop speediness is then tuned by suitable choice of the generalized time constant. Simulation results are presented to evaluate the achievements and make comparison with those of a similar method.

Keywords: Characteristic polynomial, CDM, Transient response, Bessel-Thompson filter

1. INTRODUCTION

To define the desired closed loop transient response in terms of the characteristic polynomials is one of the most important issues in the control system design. Almost all the researches in the area have discussed the matter for the special form of the closed loop transfer function i.e. all-pole transfer functions [1-3].

The relation between characteristic polynomial coefficients and transient response has been studied, for the first time in 1953, by Graham and his colleague who developed the Integral Time Absolute Error (ITAE) standard form [4]. In 1960, Kessler did some changes in the defined form to get step responses with less oscillations and overshoots [5]. His proposed form causes 8 percent overshoot. It is also more stable (has less oscillatory response) in comparison with IATE form.

The basics of the new methods in determining the characteristic polynomial coefficients are studies that Naslin has done in the 1960s [6]. One of the most important methods in the category is the Coefficients Diagram Method (CDM) which has been introduced by Manabe [7]. Although, the CDM is considered as a new method, its principles have been utilized in the industry for about 40 years [8]. The most recent method is the Characteristic Ratio Assignment (CRA) that has been introduced in [9]. This method profits by the structure used in the design of the Butterworth filter and it offers a characteristic polynomial by which a suitable transient response can be obtained.

In this note, we have proposed a new method to determine a characteristic polynomial for an all-pole system inspiring the same idea as of the CRA method. We base our design on the Bessel-Thompson filter which produces step responses with desired overshoot and speed.

The paper has been organized as follows. In Section 2, characteristic ratio is defined and adjusted to obtain desired speed specification. Tuning of the overshoot is discussed in Section 3. Conclusion of the work is derived in Section 4.

2. ADJUSTING SPEEDINESS OF THE RESPONSE

To start the discussion, we use Naslin's definitions [6]. Suppose that $p(s)$ is a Hurwitz polynomial with positive real coefficients.

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 \quad (1)$$

Characteristic ratios are defined as ($1 \le i < n$):

$$\alpha_i = \frac{a_i^{\ 2}}{a_{i-1} a_{i+1}} \quad (2)$$

Also generalized time constant is:

$$\tau = \frac{a_1}{a_0} \quad (3)$$

Having a_0, α_i s and τ, the coefficients of the polynomial are uniquely determined as:

$$a_1 = \tau\, a_0,\ a_i = \frac{\tau^i a_0}{\alpha_{i-1}\alpha_{i-2}^2\alpha_{i-3}^3 ... \alpha_1^{i-1}} \quad (4)$$

Now, suppose that $H(s)$ and $G(s)$ are transfer functions with the following forms:

$$G(s) = \frac{N(s)}{D(s)} = \frac{n_m s^m + n_{m-1} s^{m-1} + ... + n_0}{d_n s^n + d_{n-1} s^{n-1} + ... + d_0} \quad (5)$$

$$H(s) = \frac{A(s)}{B(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + ... + a_0}{b_n s^n + b_{n-1} s^{n-1} + ... + b_0} \quad (6)$$

The following theorem is stated and proved in [9]:

Theorem: If $G(s)$ and $H(s)$ produce outputs $y(t)$ and $y(\beta t)$ for the same input $r(t)$, then the coefficients of these two transfer functions are related according to the following rule:
$a_i = n_i / \beta^i$ for $i = 0, 1, \cdots, m$ and $b_j = d_j / \beta^j$ for $j = 0, 1, \cdots, n$;
With due attention to the definitions given in the beginning of the section and the above theorem, one concludes:
Result 1: If $G(s)$ is stable and minimum phase then $H(s)$ will be determined uniquely with the following relations:
$\alpha_i^A = \alpha_i^N$ for $i = 1, 2, \cdots, m-1$, $\alpha_i^B = \alpha_i^D$ for $i = 1, 2, \cdots, n-1$, $\tau^A = (1/\beta)\tau^N$, $\tau^B = (1/\beta)\tau^D$ and $a_0 / b_0 = n_0 / d_0$.

Where α_i^N, α_i^D, α_i^A and α_i^B are characteristic ratios of $N(s)$, $D(s)$, $A(s)$ and $B(s)$ respectively. Also τ^N, τ^D, τ^A and τ^B are their corresponding generalized time constants.

Example 1: Consider the following system with the characteristic polynomial coefficients that were determined based on the IATE standard for $n=5$ [10].

$$G_1(s)=\frac{1}{s^5+2.8s^4+5.0s^3+5.5s^2+3.4s+1} \qquad (7)$$

The characteristic ratios and generalized time constant of the system are:

$$[\alpha_4,\alpha_3,\alpha_2,\alpha_1]=[1.568,1.624,1.779,2.102], \quad \tau=3.4$$

Fig. 1 shows the unit step response of the system. As it is observed, the rise time is about 2.8 *sec*. Now suppose that one wants to obtain a rise time of 1 *sec* without any changes in the overshoot. With due attention to Result 1, it suffices to keep all of α_i s fixed and choose the τ as:

$$\tau=\frac{1}{2.8}\times 3.4=1.21$$

The resulted system will be as:

$$G_2(s)=\frac{1}{0.0058s^5+0.0453s^4+0.2267s^3+0.6993s^2+1.2124s+1} \qquad (8)$$

In Fig. 1, the step response of this system is also depicted. As it is expected, the rise time in this case is 1 *sec*.

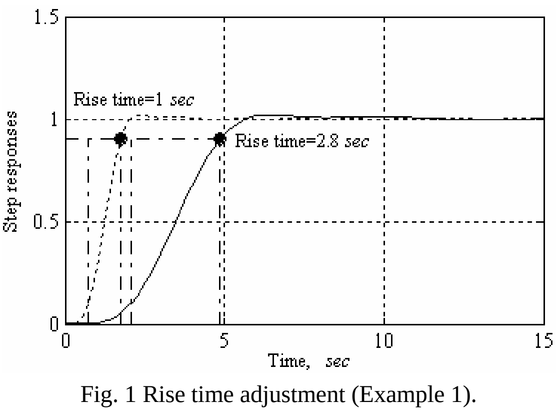

Fig. 1 Rise time adjustment (Example 1).

3. TUNING OF THE OVERSHOOT

In this part, we put the topic of our discussion on the basis of the obtained results in Chestnut's experimental studies [11]. He advocated that lower (or no) resonant peak as well as low steeper attenuation slope in the high frequency region is necessary to achieve smaller overshoot.

Here, we have taken help from Bessel-Thomson filter design method for selection a characteristic polynomial which results in desired transient response. Among filters, this filter has been known as a maximally-flat time delay [12]. The frequency response of Bessel-Thomson filter does not have any resonant peak (Fig. 2) and in comparison with the other famous filters such as Butterworth, Chebyshev, and Elliptic, its frequency magnitude in high frequencies has less slope (Fig. 3) [13]. Therefore, the frequency response of this filter adapts with the Chestnut's conditions.

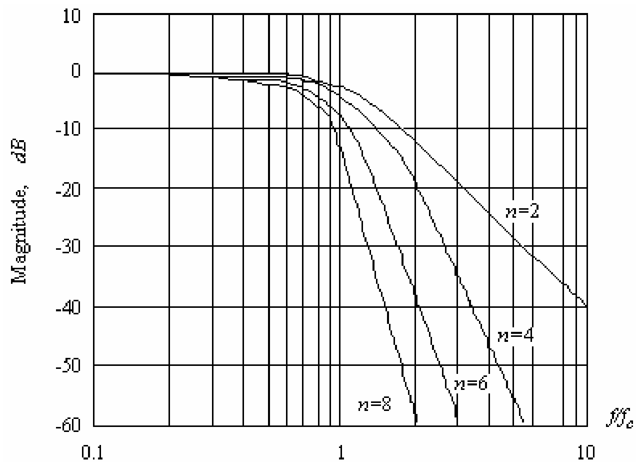

Fig. 2 Bessel-Thomson filter response for different orders.

Characteristic polynomial coefficients of Bessel-Thomson filter are taken from Bessel polynomials which have been introduced by Krall and Fink in 1948 [14]. The characteristic polynomial is defined as:

$$p(s)=a_n s^n+\ldots+a_1 s+a_0,\ a_{n-k}=\frac{(n+k)!}{2^k(n-k)!k!} \qquad (9)$$

From (2):

$$\alpha_{n-k}=(1+\frac{1}{k})(1-\frac{1}{n+k+1})(1+\frac{1}{n-k}), \quad 1\le k\le n-1 \qquad (10)$$

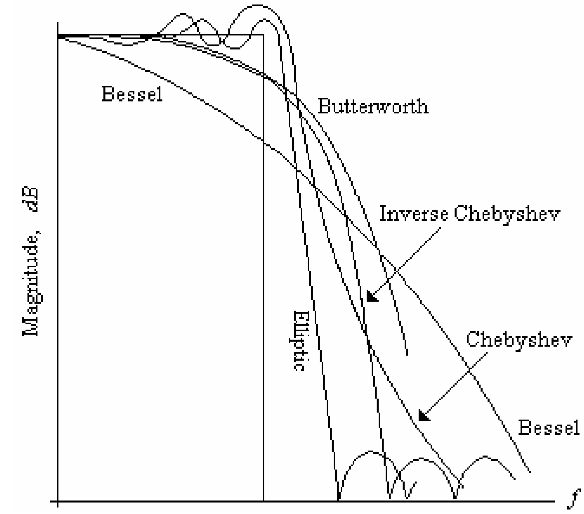

Fig. 3 Magnitude responses of standard filters.

The following relation exists among α_k s:

$$\prod_{k=1}^{n-1}\alpha_k=\frac{n(n+1)}{2} \qquad (11)$$

Relation in (11) is easily proved by simple replacing of α_k s from (10).

We know that the overshoot for the Bessel-Thomson filter is less than 1%. Hence, If we choose α_k 's according to (10), the response will have overshoot less than 1%. The following algorithm is proposed to determine a characteristic polynomial that results in an adjustable overshoot along with the required speed:

Algorithm:

1- *Select the generalized time constant arbitrarily ($\tau = \tau_1$).*
2- *Choose an appropriate degree for the characteristic polynomial (n).*
3- *Calculate α_i s according to (10).*
4- *Adjust the generalized time constant ($\tau = \tau_2$) in order to get a response with demanded speed (Result 1).*

For example if the purpose is to reach a settling time of t_s and the resulted response at stage 3 of the algorithm has the setting time of t'_s, then τ_2 should be determined as:

$$\tau_2 = \frac{t_s}{t'_s} \times \tau_1 \tag{12}$$

The characteristic polynomial which is deduced from the above algorithm gives us a response with very low overshoot. If one decides to obtain lower overshoot or responses without overshoot, this algorithm should be modified to satisfy Chestnut's criteria. We try to decrease the slope of the transfer function magnitude at high frequencies by applying the following modifications. α_{n-1} is replaced by $\gamma\alpha_{n-1}$ ($0 < \gamma \le 1$) and α_k s ($1 \le k \le n-2$) are divided by $\sqrt[n-2]{\gamma}$. These changes still satisfy (11). In other words, we propose the following characteristics ratio.

$$\begin{aligned} \alpha_{n-1} &= 2\gamma \frac{n+1}{n+2} . \frac{n}{n-1} \\ \alpha_k &= (1+\frac{1}{k})(1-\frac{1}{n+k+1})(1+\frac{1}{n-k})(\sqrt[n-2]{\gamma})^{-1} \end{aligned} \tag{13}$$

Decrease of γ, reduces the slope of the transfer function magnitude in the high frequencies. Fig.4 shows the effect for $n = 7$ and three different γ. The simulation results show that one can reach to an almost 0% overshoot by decreasing γ. According to [15], we know that if the curvature of the a_i become larger, the system become more stable. Decrease of γ also increase the system relative stability. Fig. 5 shows the point for $n = 8$, $\tau = 6$ and three different γ. Table 1 shows bandwidth, gain margin, phase margin, overshoot, settling time, rise time and tracking error according to integral error indices criteria for three different γ. As it is observed decrease in γ produces larger gain and phase margin. The algorithm is then modified as follows.

Modified Algorithm:

1- *Select the generalized time constant arbitrarily ($\tau = \tau_1$).*
2- *Choose an appropriate degree for the characteristic polynomial (n).*
3- *Set γ less than one and then calculate α_i s from (13). This selection will result in lower overshoot. A specific result can be achieved by few trial and error iterations.*
4- *Adjust the generalized time constant ($\tau = \tau_2$) in order to get a response with demanded speed (Result 1).*

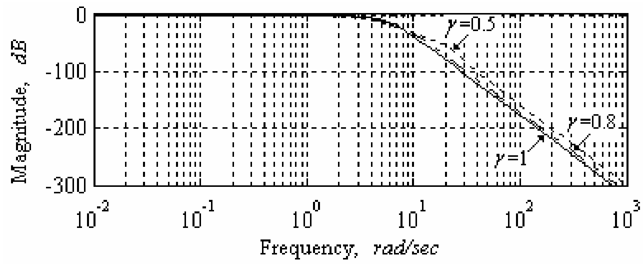

Fig. 4 Decrease in transfer function magnitude slope by decrease in γ.

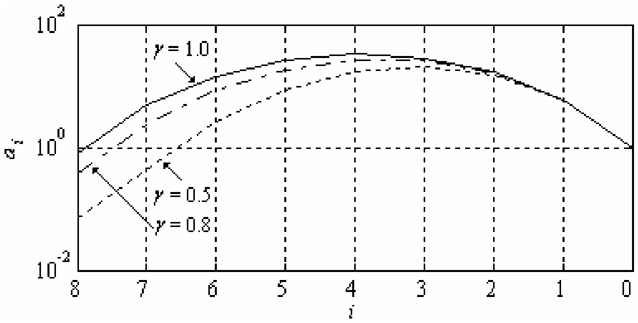

Fig. 5 Increase in system relative stability by decrease in γ.

Table 1. Bandwidth, Gain and Phase margins, Overshoot, Settling and Rising times and error amounts for three different γ

	$\gamma = 1$	$\gamma = 0.9$	$\gamma = 0.8$
Bandwidth	2.9453 *Hz*	2.6802 *Hz*	2.4667 *Hz*
Gain margin	2.4838	2.6352	2.8116
Phase margin	62.7480	63.4181	64.1382
Overshoot	0.49%	0	0
Settling time	1.5659 *sec*	1.6827 *sec*	1.7997 *sec*
Rise time	0.7413 *sec*	0.8077 *sec*	0.8680 *sec*
Error (IAE)	0.9604	0.9898	0.9827
Error (IATE)	0.4848	0.5297	0.5272
Error (ISE)	0.8356	0.8246	0.8082
Error (ITSE)	0.3711	0.3662	0.3550

Example2: We design the controller for the same system as in [9] using the above algorithm. Our purpose is to control the system such that the closed loop step response has no overshoot and 1% settling time of 1 sec. Transfer function of the system is:

$$G(s) = \frac{n(s)}{d(s)} = \frac{600(s^2+s+2.5)}{s^6+4.8s^5+12.84s^4+11.232s^3 +65.02s^2+213.27s+1189.3} \tag{14}$$

We use a two degree of freedom controller structure [16]. Fig. 6 shows the block diagram of the closed loop system. Consider the following polynomials.

$$\begin{aligned} a(s) &= a_5 s^5 + ... + a_1 s + a_0 \\ b(s) &= b_5 s^5 + ... + b_1 s + b_0 , \quad f(s) = f_0 \end{aligned} \tag{15}$$

The following assumptions are considered to get a zero-free closed loop system.

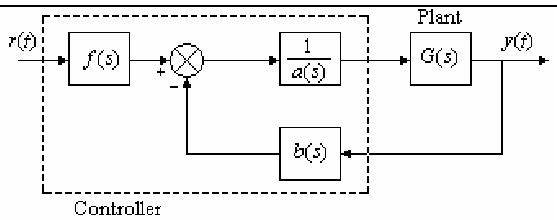

Fig. 6 Two degree of freedom control configuration.

$$a(s) = \bar{a}(s)(s^2 + s + 2.5),\ n(s) = \bar{n}(s)(s^2 + s + 2.5) \qquad (16)$$

The closed loop transfer function can be written as follows:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{f(s)n(s)}{p(s)} \qquad (17)$$

Where $p(s)$ is the characteristic polynomial.

$$p(s) = (\bar{a}(s)d(s) + b(s)\bar{n}(s))(s^2 + s + 2.5) = \bar{p}(s)(s^2 + s + 2.5) \qquad (18)$$

$\bar{p}(s)$ is of order 9 and is determined based on the proposed algorithm. First, we set $\tau = 1$. Then derive the step responses for two values of γ ($\gamma = 1$ and $\gamma = 0.7$). When $\gamma = 1$ the response has an overshoot of 0.217% and when $\gamma = 0.7$ the overshoot is almost zero (Fig. 7). The settling time of non overshooting response is equal to 1.916 *sec*. Therefore we adjust τ as:

$$\tau = \frac{1}{1.916} \times 1 = 0.5219$$

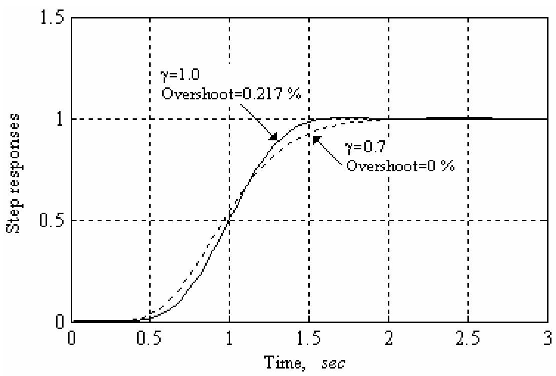

Fig. 7 Step response of the desired closed loop systems with various γ s (Example 2).

Fig. 8 shows the response for the final closed loop system. The controller polynomials are given as follows:

$$a(s) = 1.8021 \times 10^{-5} s^5 + 0.0015 s^4 + 0.0878 s^3 + 3.0639 s^2 + 3.1897 s + 7.4348 \qquad (19\text{-}1)$$

$$b(s) = 0.1112 s^5 + 2.1852 s^4 + 25.0559 s^3 + 182.3752 s^2 + 781.6529 s + 1494 \qquad (19\text{-}2)$$

$$f(s) = 1500 \qquad (19\text{-}3)$$

Example 3: Through this example we try to compare our proposed method with the one called CRA and was introduced in [9]. Assume that the desired response has 2% settling time of 1 *sec* and overshoot less than 0.1%.

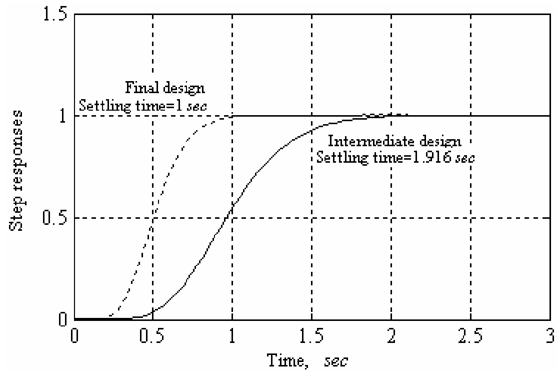

Fig. 8 Step response of the desired closed loop systems with various τ s and γ =0.7 (Example 2).

Table 2 shows bandwidth, rise time and tracking error according to integral error indices criteria. As it is observed the proposed algorithm possesses larger bandwidth and lower rise time than the CRA method. However, the CRA method produces less error.

Now, we study the sensitivity of the response characteristics (overshoot and rising time) with respect to variations in the coefficients. Overshoot and rise time changes caused by ±50% variations in a_1 and a_5 are given in Table 3. We observe that the CRA method indicates less sensitivity to coefficients variations in some cases. However, the differences are negligible.

Table 2. Bandwidth, rise time and error amounts for presented algorithm and CRA algorithm

	New Algorithm	**CRA Algorithm**
Bandwidth	4.1442 *Hz*	4.0314 *Hz*
Rise time	0.5210 *sec*	0.5349 *sec*
Error (IAE)	0.5298	0.5012
Error (IATE)	0.1590	0.1417
Error (ISE)	0.4185	0.3965
Error (ITSE)	0.0978	0.0890

Table 3. Overshoot and rise time variations with respect to 50% changes in the coefficients

Coefficient Variation	**Method**	**+50% Variation**	**-50% Variation**
a_1	New Algorithm	O.S=0 t_s =2.62 *sec*	O.S=52.7% t_s =6.11 *sec*
	CRA Algorithm	O.S=0 t_s =2.56 *sec*	O.S=49.6% t_s =5.12 *sec*
a_5	New Algorithm	O.S=0.63% t_s =1.04 *sec*	O.S<0.1% t_s =0.95 *sec*
	CRA Algorithm	O.S=0.23% t_s =1.01 *sec*	O.S<0.1% t_s =0.98 *sec*

4. CONCLUSION

In this paper, a new algorithm is introduced which is based on the Bessel-Thompson filter design method. By applying this algorithm, a characteristic polynomial which possesses the desired overshoot and speediness can be determined. If an open loop system is minimum phase, we can use the controller design method –as was used in Example 2- for closed loop system to get the desired characteristic polynomial.

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REFERENCES

[1] S. Jayasuriya and J. W. Song, "On the synthesis of compensators for non overshooting step response", *Proc. Amer. Control Conf*, pp. 683-684, 1992.G.-D. Hong, "Linear controllable systems," *Nature*, Vol. 135, pp. 18-27, 1990.

[2] R.H. Middleton and S. F. Graebe, "Slow stable open-loop poles: To cancel or not to cancel", *Automatica*, **35**, pp. 877-886, 1999.

[3] G.C. Goodwin, A. R. Woodyatt, R. H. Middleton and J. Shim, "Fundamental limitations due to –axis zeros in SISO systems", *Automatica*, **35**, 857-863, 1999.

[4] D. Graham, and R.C. Lathrop, "The synthesis of optimum transient response: criteria and standard forms", *AIEE Transactions*, **72**(**II**), pp. 273-288, 1953.

[5] C. Kessler, "Ein Beitrag zur Theorie Mehrschleifiger Regulation", *Regelungst*, **8**(**8**), pp. 261-266, 1960.

[6] P. Naslin, *Essentials of optimal control*. MA: Boston Technical Publishers, Cambridge, 1969.

[7] S. Manabe, "Coefficient diagram method", In *Proc. 14th IFAC Symp. Automatic Control Aerospace*, Seoul, Korea, pp. 199-210, 1998.

[8] S. Manabe and Y.C. Kim, "Recent development of Coefficient Diagram Method", *ASSC'2000 3rd Asian Control Conference*, Shanghai, 2000.

[9] Y.C. Kim, L.H. Keel and S.P. Bhattacharyya, "Transient response control via characteristic ratio assignment", *IEEE Trans. Automat. Contr.*, **48,** pp. 2238-2244, 2003.

[10] R.C. Dorf and R.H. Bishop, *Modern Control Systems*, 8th Ed. Addison Wesley Longman, Menlo Park, CA, 1998.

[11] H. Chestnut and R.W. Mayer, *Servomechanisms and Regulating System Design*, **I**, New York, Wiley, 1959.

[12] L.P. Huelsman, Active and Passive Analog Filter Design, McGraw-Hill, 1993.

[13] S. Winder, *Analog and Digital Filter Design*, 2nd Ed., Newnes, 2002.

[14] H.L. Krall and O. Fink, "A New Class of Orthogonal Polynomials: The Bessel Polynomials", *Trans. Amer. Math Soc.*, **65**, pp. 100-115, 1948.

[15] S. Manabe, "Brief Tutorial and Survey of Coefficient Diagram Method", 4th *Asian Control Conference*, Singapore, 2002.

[16] C.T. Chen, Analog and Digital Control System Design: Transfer Function, State Space and Algebraic Methods. Ft. Worth, TX: Saunders College Publishing, 1993.