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Global Bifurcations and Chaos Via Breaking of KAM Tori of an Harmonically Excited Imperfect Circular Plate.

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ABSTRACT

Global bifurcations and chaos in modal interactions of an imperfect circular plate with one-to-one internal resonance are investigated. The case of primary resonance, in which an excitation frequency is near natural frequencies, is considered. The damping force is not included in the analysis. The renormalization-group technique for KAM tori is used to obtain the criteria for large-scale stochasticity in the system.

1. Introduction

Dynamical systems having two of their linear natural frequencies nearly equal exhibit complicated and interesting phenomena when nonlinear terms are taken into account. Plates may be one of the systems. The dynamics of plates related with non-linear modal interactions was widely described in terms of local bifurcations by Sridhar, Mook and A.H. Nayfeh [1], Yang and Sethna. [2]. Lee et al. [3, 4] studied modal interactions of a circular plate and of the plate on an elastic foundation [5, 6] for which internal resonance occurs. Global bifurcations have been examined for a wide class of problems. Feng and Sethna [7] studied global bifurcations of a Hamiltonian system with a certain symmetry in terms of breaking of heteroclinic orbits. Yeo and Lee [8] studied global bifurcations in modal interactions of a circular plate using a method developed by Wiggins and Kovačič [9]. They investigated homoclinic orbit created in a resonance resulting from perturbation. Lee and Samoylenko [10] extended Yeo and Lee's work [8] to investigate breaking of heteroclinic orbits created in a non-resonance case for undamped system by use of Melnikov method [11].

One of the key mechanisms of the transition to chaos in Hamiltonian systems is the breaking of invariant tori [12, 13]. In this work chaos due to breaking of KAM tori is considered. The renormalization-group analysis [14,15] is used to study the appearance of large-scale stochasticity in modal interactions of a circular plate.

2. Governing equations

The equations governing the free, undamped oscillations of non-uniform circular plates were derived by Efstathiades [16]. Yeo and Lee [8] simplified these equations to fit the special case of non-uniform circular plate, for which forcing terms were added.

Assume that transverse displacement of the plate could be expressed as a combination of two linearized modes. Neglecting damping terms we reduce equations obtained by Yeo and Lee as follows:

$$\ddot{x}_i + \omega_i^2 x_i + \varepsilon\gamma\omega_i^2 x_i (x_1^2 + x_2^2) = \varepsilon\mu_i \cos\lambda t,\ i = 1,2 ,$$

where x_i are amplitudes of normal modes, ω_i are normal frequencies, γ plays a role of the parameter of nonlinearity, μ_i are amplitudes of excitation and ε is a small parameter.

We consider external resonance $\omega_1 = \lambda$. In order to consider internal resonance due to imperfection, $\omega_1 \approx \omega_2$, we introduce internal detuning parameter β as follows: $\omega_2 = \omega_1 + \varepsilon\beta$.

By use of the method of multiple scales [17] we get solvability conditions

$$Z'_1 = i[\varepsilon F_1 + 3\gamma\lambda Z_1^2 Z_1^* / 2 + \gamma\lambda Z_1 Z_2 Z_2^* + \gamma\lambda Z_2^2 Z_1^* / 2] \quad (1a)$$

$$Z'_2 = i[\varepsilon F_2 + \beta Z_2 + 3\gamma\lambda Z_2^2 Z_2^* / 2 + \gamma\lambda Z_2 Z_1 Z_1^* + \gamma\lambda Z_1^2 Z_2^* / 2]. \quad (1b)$$

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Here $F_j = -\mu_j / 4\varepsilon\lambda$ are forcing terms and Z_j is a slow-time amplitude:

$$x_{j0} = Z_j(T_1)e^{i\lambda T_0} + Z_j^*(T_1)e^{-i\lambda T_0}, \quad j = 1,2,$$
$$T_k = \varepsilon^k t, \quad k = 0,1.$$

Asterisk and prime denote, respectively, complex conjugate and a differentiation with respect to slow time.

Assuming harmonic amplitudes Z_j as $Z_j = \sqrt{2a_j}(\sin\phi_j + i\cos\phi_j)$, and making variable change with rescaling:

$$\begin{aligned} P_1 &= a_1\gamma\lambda, & Q_1 &= \phi_1 - \phi_2 \\ P_2 - P_1 &= a_2\gamma\lambda, & Q_2 &= \phi_2 - \pi/2 \\ F_1 &= -f_1/\sqrt{\gamma\lambda}, & F_2 &= f_2/\sqrt{\gamma\lambda} \end{aligned}$$

we transform the system (1) to a Hamiltonian form with Hamiltonian

$$\begin{aligned} H = H_0 + \varepsilon H_1 &= \beta(P_1 - P_2) \\ &- (1+\cos 2Q_1)P_1(P_1 - P_2) - 3P_2^2/2 \\ &- \varepsilon\left[\frac{f_1}{\sqrt{2P_1}}\sin(Q_1+Q_2) + \frac{f_2}{\sqrt{2(P_2-P_1)}}\cos Q_2\right]. \end{aligned} \tag{2}$$

In order to have real-valued function H we must require $0 \le P_1 \le P_2$. In the absence of perturbation i.e. $\varepsilon = 0$, system (2) is a completely integrable Hamiltonian system with $P_2 = P_{20}$ as a conserved quantity. For $-P_{20}/2 < \beta < P_{20}/2$ the unperturbed system has heteroclinic orbits [10].

3. Invariant Tori and Renormalization Operator.

In order to investigate invariant tori of system H_0 we construct the action-angle coordinates. Existence of heteroclinic orbits does not allow making a global action-angle transformation. It is, however, possible to construct the action-angle transformation locally. The actions are given by:

$$I_1 = I_1(P_{20}, \beta, h), \quad I_2 = P_2, \tag{3}$$

where h is an energy level of the system H_0. Expression $I_1(P_{20}, \beta, h)$ is given in an integral form. The domain in coordinates (I_1, I_2), for which local transformation (3) is valid could be determined by relation $0 \le P_1 \le P_{20}$. It gives the critical values of energy h:

$$h_1 = H_0\big|_{P_1=0}, \quad h_2 = H_0\big|_{P_1=P_{20}},$$

which determine the range of possible energy values

$$h_1 \le h \le h_2, \quad if \ \beta < 0$$
$$h_2 \le h \le h_1, \quad if \ \beta > 0.$$

This range gives conditions on possible values of I_2 for given value of energy h:

$$I_2^{\min} \le I_2 \le I_2^{\max}, \tag{4}$$

where upper and lower boundaries of I_2 are functions of β, P_{20} and h.

The phase space of the unperturbed system (2) is foliated by 2-dimensional invariant tori. In action-angle coordinates each torus is specified by two frequencies: $\omega_1 = \partial h/\partial I_1$, $\omega_2 = \partial h/\partial I_2$. The ratio $W = \omega_1/\omega_2$ is said to be the *winding number* of a torus. We will refer to an invariant torus of rational winding number $W = p/q$ as *resonant invariant torus* that corresponds to a *resonance* denoted by a pair *(p,q)*. An invariant torus of irrational winding number is said to be *non-resonant*, or *irrational.* Actions (3) allow us to calculate winding ratio of a given invariant torus.

A rational invariant torus is foliated by an infinite number of periodic orbits. The Poincaré -Birkhoff theorem [18] states that only a finite number of them survive under perturbation. Half of these periodic orbits are elliptic, and the other half hyperbolic. In the neighborhood of the stable and unstable manifold of hyperbolic orbits chaotic motion could be observed. We call this phenomenon breaking of resonant torus.

The behavior of irrational invariant tori can be studied using the Kolmogorov-Arnold-Moser theorem [12]. This theorem states that for a near-integrable nondegenerate Hamiltonian system under small perturbation irrational invariant tori with Diophantine frequency vectors do not decay but are only deformed. We refer to the tori from the KAM theorem as *KAM tori.* Under sufficient perturbation, however, KAM tori are also destroyed. Destruction of all KAM tori in a certain region of phase space lead to *large-scale stochasticity* in this region also called *global chaos* [13, 15].

Study of Poincaré sections of the system flow shows resonant tori, which break first under perturbation. We refer to these tori and corresponding resonances as *main resonant tori* and *main resonances* respectively. For

$\beta > 0$ two main resonances are (2, 1) and (3, 1). Corresponding resonant tori have winding numbers 2 and 3. For $\beta < 0$ main resonances are (0, −1) and (1, −1). Main resonant tori that correspond to these resonances have winding numbers 0 and −1.

Positions of main resonant tori depend on value of parameters β, P_{20} and energy h. For some values of parameters main resonant tori may lie outside of ranges given by inequality (4). Solving the equations on the winding number $W(I_2^{\min}) = w$ and $W(I_2^{\max}) = w$ for $w = 2,3,0$ and −1 we obtain regions in parameter space (β, P_{20}), where main resonant tori exist.

Using the method suggested by Pronine [15] we find the winding number W_{cr} of the KAM torus which will be the last invariant torus destroyed under perturbation between two given resonant tori. For $\beta < 0$ the last KAM torus has $W_{cr} = -g$ and for $\beta > 0$ $W_{cr} = 2 + g$, where $g = (\sqrt{5} - 1)/2$ is the golden mean. Any irrational number has a unique representation as an infinite continued fraction [19]. Truncation of a continued fraction gives a rational approximation of the irrational number. Winding numbers of the main resonant tori correspond to first two rational approximants of irrational winding number W_{cr}.

In order to get a threshold for destruction of the last KAM torus between two main resonances $\mathbf{m}$ and $\mathbf{p}$ a renormalization operator was constructed. It utilizes the Hamiltonian in the normal form:

$$H_n = \omega I_1 + I_2 + aI_1^2 + 2bI_1I_2 + cI_2^2 + M\cos\theta_1 + P\cos\theta_2 \qquad (5)$$

with $0 < \omega < 1$, $a > 0$ and $a^2 + b^2 + c^2 = 1$. System (2) could be approximated by the normal form (5). The coefficients ω, a, b and c depend on the Hamiltonian H_0, the energy h and the considered KAM torus. Coefficients M and P represent harmonics of H_1 which are resonant to frequencies of tori corresponding to $\mathbf{m}$ and $\mathbf{p}$.

The threshold to large-scale stochasticity was calculated in parameter space (β, P_{20}). Values of the small parameter ε and excitation amplitudes f_1 and f_2 were fixed ($\varepsilon = 0.05$, $f_1 = f_2 = 1.0$). The critical values of parameters for which the last KAM torus is destroyed were found by use of bisection method along lines $P_{20} / \beta = const$. The order of accuracy of the bisection method was 10^{-4}. Computations were done for five values of energy between h_1 and h_2. The union of chaotic regions in parameter space obtained for the range of energies gives the region where large-scale stochasticity may be observed for some value of energy, which is available for the system.

The picture of global bifurcations of considered system is presented on Fig 1. In region 1 the system has main resonances (0, −1) and (1, −1). Breaking of corresponding resonant tori under perturbation lead to stochastic layers. In the subregion 1a stochastic layers formed by two main resonant tori merge and large-scale stochasticity occurs. On the boundary between regions 1 and 1a the last KAM torus with winding number $-g$ is destroyed. In region 2 the system has main resonant tori corresponding to resonances (2, 1) and (3, 1). Under perturbation this tori break and form stochastic layers. There is a subregion denoted as 2a where large-scale stochasticity takes place. On the boundary between regions 2 and 2a the last KAM torus with winding number $2 + g$ is destroyed. In regions 3, 4 and 5 system has no resonances (2, 1), (3, 1), (0, −1), or (1, −1). Nevertheless, chaos may appear in these regions due to breaking of other resonant tori. These tori correspond to resonances of irrational tori, which are more stable then KAM tori described above. Therefore, for considered excitation intensity the area of stochastic layers in regions 3,4 and 5 will be small, compared to area of stochastic layers appearing in regions 1 and 2.

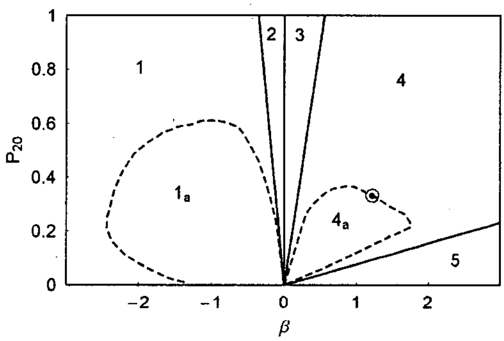

Fig. 1. The global bifurcation diagram.

In order to illustrate the meaning of a boundary between regions 1-1a and 2-2a a Poincaré map was calculated for point marked by a circle in Fig. 1. This map is presented in Fig. 2. Here we see the last KAM-torus with winding ratio $2 + g$ surrounded by stochastic layers and destroyed rational tori.

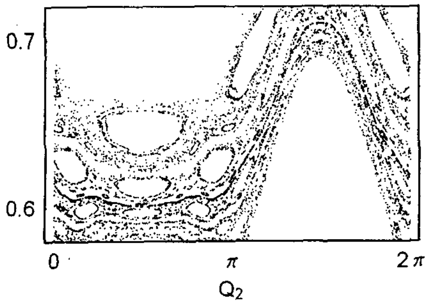

Fig. 2. The last KAM torus with winding ratio $2+g$.
($\beta = 1.2163$, $P_{20} = 0.3314$, $\varepsilon = 0.05$, $f_1 = f_2 = 1$.)

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