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상대절점좌표를 이용한 비선형 유한요소해석법

A Relative for Finite Element Nonlinear Structural Analysis

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ABSTRACT

Nodal displacements are referred to the initial configuration in the total Lagrangian formulation and to the last converged configuration in the updated Lagrangian formulation. This research proposes a relative nodal displacement method to represent the position and orientation for a node in truss structures. Since the proposed method measures the relative nodal displacements relative to its adjacent nodal reference frame, they are still small for a truss structure undergoing large deformations for the small size elements. As a consequence, element formulations developed under the small deformation assumption are still valid for structures undergoing large deformations, which significantly simplifies the equations of equilibrium. A structural system is represented by a graph to systematically develop the governing equations of equilibrium for general systems. A node and an element are represented by a node and an edge in graph representation, respectively. Closed loops are opened to form a spanning tree by cutting edges. Two computational sequences are defined in the graph representation. One is the forward path sequence that is used to recover the Cartesian nodal displacements from relative nodal displacements and traverses a graph from the base node towards the terminal nodes. The other is the backward path sequence that is used to recover the nodal forces in the relative coordinate system from the known nodal forces in the absolute coordinate system and traverses from the terminal nodes towards the base node. One closed loop structure undergoing large deformations is analyzed to demonstrate the efficiency and validity of the proposed method.

1. Introduction

Geometrically nonlinear analyses have been investigated by many researchers [1-4]. Their equations of equilibrium are based on either the total Lagrangian formulation or the updated Lagrangian formulation. Avello referred kinematic variables relative to the initial configuration and he expressed the strains in a moving frame [5]. Therefore, the strains were invariant for finite rigid body deformations. Shabana presented an absolute nodal coordinate formulation for flexible multibody dynamics [6,7]. Shimizu considered the rotary inertia effects [8]. Bae has generalized a recursive formulation for the rigid body dynamics [9,10]. The recursive formulation has been further developed for the flexible body dynamics and design sensitivity analysis [11,12]. In this research, the recursive formulation is applied to solve the geometric nonlinear problems in truss structures undergoing large deformations. The proposed formulation employs the moving reference frame approach, which was proposed [13,14]. A moving reference frame is introduced to represent a finite rigid body motion. Deformation at a point of a flexible body was super-imposed on the rigid body motion.

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2. Kinematics of relative nodal displacement

2.1 Graph theoretic representation of a structure

This paper proposes a relative nodal displacement method in formulating the equations of equilibrium. Since the absolute nodal displacements are obtained by accumulating the relative nodal displacements along a path, element connectivity information must be identified prior to generating the equations of equilibrium for a truss structure. Therefore, the topology analysis must be carried out for a structural system discretized into many finite elements. The discretized systems can be represented by a graph [10]. If a structure possesses a loop in its graph theoretic representation, it is called as a closed loop system. Otherwise, it is called as an open loop system.

2.2 Kinematic definitions

Consider a system consisting of two beam finite elements as shown in Fig. 1.
Nodes $i-1$ and i are assumed to be inboard nodes of nodes i and $i+1$ in a graph, respectively. The $X-Y-Z$ is the inertial reference frame and $x_k - y_k - z_k$ is the nodal reference frame attached to a node k, and $\mathbf{r}_k$ is a position vector of the node k.

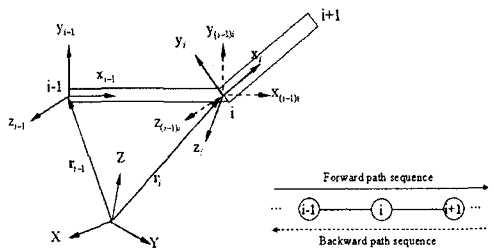

Fig. 1 Two finite beam elements and its graphic representation

The $x_{(i-1)i} - y_{(i-1)i} - z_{(i-1)i}$ is the reference frame attached to a node i and the first subscript $i-1$ denotes the inboard node number of the second subscript i. The relative nodal displacements measured in its inboard nodal reference frame are solved in research. The generalized coordinates for the relative nodal position and orientation displacements of a node are denoted by $\mathbf{u}'_{(i-1)i}$ and $\mathbf{\Theta}'_{(i-1)i}$, respectively. The nodal position and orientation of node i in the inertial reference frame can be expressed in terms of these of node $i-1$ and the relative nodal displacements as follows:

$$\mathbf{r}_i = \mathbf{r}_{(i-1)} + \mathbf{A}_{(i-1)}\left(\mathbf{s}'_{(i-1)i0} + \mathbf{u}'_{(i-1)i}\right) \tag{1}$$

$$\mathbf{A}_i = \mathbf{A}_{i-1}\mathbf{D}_{(i-1)i}(\Theta'_{(i-1)i})\mathbf{C}_{(i-1)i} \tag{2}$$

In Eqs. (1) and (2), $\mathbf{A}_k$ denotes the orientation matrix for node k in the initial reference frame, $\mathbf{C}_{(i-1)i}$ denotes the constant transformation matrix from $x_i - y_i - z_i$ to $x_{(i-1)i} - y_{(i-1)i} - z_{(i-1)i}$, $\mathbf{s}'_{(i-1)i0}$ denotes the position vector of node i measured in $x_{(i-1)} - y_{(i-1)} - z_{(i-1)}$ in undeformed state, and $\mathbf{u}'_{(i-1)i}$ denotes the deformation vector of node i relative to the nodal frame $i-1$. $\mathbf{D}_{(i-1)i}$ is the transformation matrix due to a rotational displacement of $x_{(i-1)i} - y_{(i-1)i} - z_{(i-1)i}$ relative to the nodal frame $i-1$.

Taking a variation of Eqs. (1) and (2) yield

$$\delta\mathbf{Z}_i = \mathbf{B}_{(i-1)i1}\delta\mathbf{Z}_{(i-1)} + \mathbf{B}_{(i-1)i2}\delta\mathbf{q}_{(i-1)i} \tag{3}$$

where

$$\delta\mathbf{Z}_k = \left[\delta\mathbf{r}_k^T \quad \delta\,\boldsymbol{\pi}_k^{'T}\right]^T \ (k = i-1, i) \tag{4}$$

$$\delta\mathbf{q}_{(i-1)i} = \left[\delta\mathbf{u}_{(i-1)i}^{'T} \quad \delta\mathbf{\Theta}_{(i-1)i}^{'T}\right]^T \tag{5}$$

$$\mathbf{B}_{(i-1)i1} = \begin{bmatrix} \mathbf{A}_{(i-1)i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{(i-1)i}^T \end{bmatrix} \begin{bmatrix} \mathbf{I} & -(\tilde{\mathbf{s}}'_{(i-1)i0} + \tilde{\mathbf{u}}'_{(i-1)i}) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{6}$$

$$\mathbf{B}_{(i-1)i2} = \begin{bmatrix} \mathbf{A}_{(i-1)i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{(i-1)i}^T \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{H}_{(i-1)i} \end{bmatrix} \tag{7}$$

Eq. (3) can be expressed in a compact matrix form as follows.

$$\delta\mathbf{Z} = \mathbf{B}\delta\mathbf{q} \tag{8}$$

3. Governing Equations of Equilibrium

3.1 Strain Energy

The strain energy in a finite element having multiple nodes is affected only by the relative nodal displacements relative to the inboard nodal frame of the element and is free from its rigid body motion. As a result, the variational form of the strain energy for a system can be obtained in a summation form as

$$\delta W = \sum_{k=1}^{n} \delta\mathbf{q}_{(k-1)k}^T \mathbf{K}_{(k-1)k}\mathbf{q}_{(k-1)k} = \delta\mathbf{q}^T\mathbf{K}\mathbf{q} \tag{9}$$

Since the stiffness matrix is generated in the nodal reference frame, the strain energy due to a rigid body motion of a node does not appear in Eq. (9). The element stiffness matrix $\mathbf{K}_{(k-1)k}$ is contributed from linear and nonlinear terms as [3]

$$\mathbf{K}_{(k-1)k} = \mathbf{K}_{(k-1)k}^L + \mathbf{K}_{(k-1)k}^{nL} \tag{10}$$

In Eq. (10), $\mathbf{K}_{(k-1)k}^L$ denotes a linear stiffness matrix, $\mathbf{K}_{(k-1)k}^{nL}$ denotes a nonlinear stiffness matrix.

3.2 External force

The virtual work done by both nodal forces $\mathbf{Q}$ described in the absolute nodal coordinate system and $\mathbf{R}$ described in the relative nodal coordinate system is obtained as follows:

$$\delta W = \delta\mathbf{Z}^T\mathbf{Q} + \delta\mathbf{q}^T\mathbf{R} \tag{11}$$

where $\delta\mathbf{Z}$ must be admissible for the kinematic relationship between $\delta\mathbf{Z}$ and $\delta\mathbf{q}$. Substitution of $\delta\mathbf{Z} = \mathbf{B}\delta\mathbf{q}$ into Eq. (11) yields

$$\delta W = \delta\mathbf{q}^T\left(\mathbf{B}^T\mathbf{Q} + \mathbf{R}\right) = \delta\mathbf{q}^T\mathbf{Q}^* \tag{12}$$

3.3 Constraint

A nodal displacement is measured relative to its inboard nodal frame in the proposed method. The relative nodal displacement can be defined only in structures having a tree topology. Therefore, if a structural system has a closed loop, it must be opened to form the tree topology. A node in a closed loop is removed and the corresponding cut constraint equations of $\mathbf{\Phi}(\mathbf{q})$ are introduced to compensate for the removed node.

3.4 Equations of equilibrium

For a closed loop system, the relative nodal displacement $\mathbf{q}$ is not independent, and the $\mathbf{q}$ must satisfy the constraint as follows.

$$\mathbf{\Phi}(\mathbf{q}) = \mathbf{0} \tag{13}$$

Taking variation of the constraint equation yields

$$\delta\mathbf{\Phi} = \mathbf{\Phi}_{\mathbf{q}}\delta\mathbf{q} = \mathbf{0} \tag{14}$$

The Lagrange multiplier theorem can be applied to obtain the following equations of equilibrium for a constrained system:

$$\delta\mathbf{q}^T\left(\mathbf{Kq} - \mathbf{Q}^* + \mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}\right) = 0 \tag{15}$$

Since the $\delta\mathbf{q}$ is arbitrary, its coefficient must be zero, which yields

$$\mathbf{F}(\mathbf{q},\lambda) = \mathbf{Kq} + \mathbf{\Phi}_{\mathbf{q}}^T\lambda - \mathbf{Q}^* = \mathbf{0} \tag{16}$$

The unknown variables of $\mathbf{q}$ and $\boldsymbol{\lambda}$ can be obtained by solving Eqs. (13) and (16) simultaneously. The unknown variables can be solved by using Newton-Raphson method as

$$\begin{bmatrix} \mathbf{F}_{\mathbf{q}} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q} \\ \Delta\lambda \end{bmatrix} = -\begin{bmatrix} \mathbf{F} \\ \mathbf{\Phi} \end{bmatrix} \tag{17}$$

where $\mathbf{F}_{\mathbf{q}} = \mathbf{K} + \left(\mathbf{\Phi}_{\mathbf{q}}^T\lambda - \mathbf{Q}^*\right)_{\mathbf{q}}$. By solving Eq. (17), the improved solution of $\mathbf{q}$ for the next iteration can be obtained as follows:

$$\mathbf{q} = \mathbf{q} + \Delta\mathbf{q} \tag{18}$$

By using Eqs. (17) and (18), the iteration continues until the solution variance remains within a specified allowable error tolerance.

4. Numerical Examples

Fig. 2 shows a closed loop system subjected to a concentrated force $\mathbf{F}$ and moment M at a point P. When $\mathbf{F}$ =[3×10^4 -3×10^4]^T [N] and M =3.0×10^4 [N · m] are applied at the point P, the deformed shape of the system is shown in the left of Fig. 3.

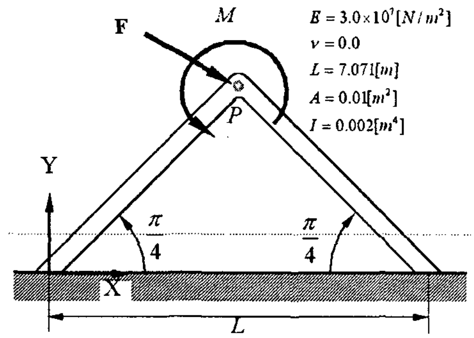

Fig. 2 A closed loop system subjected to a concentrated force and moment

It shows that the numerical results obtained by the proposed method with 20 elements and a commercial program ANSYS are almost identical. While the proposed method converges after 7 iterations, ANSYS converges after 12 iterations. When extreme loads $\mathbf{F}$ =[5×10^4 -5×10^4]^T and M =5.0×10^4[N · m] are applied at the point P, the deformed shape of the system is shown in the right of Fig. 3. While the proposed method converges after 11 iterations, ANSYS did not converge. The numerical result indicates that as the geometric nonlinearity becomes severe, the proposed method performs much better than the conventional method.

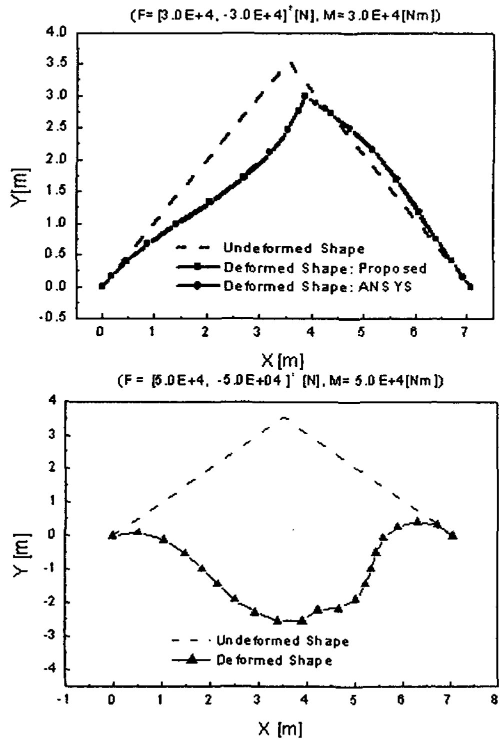

Fig. 3 Comparison of deformed shapes of the closed system

5. Conclusions

A geometric nonlinear formulation for truss structures undergoing large deformations is investigated in this research. Nodal displacements in the proposed method are referred to its adjacent nodal reference frame. Since the nodal displacements are measured relative to its inboard nodal frame, quantity of the nodal displacements is still small for a structure undergoing large deformations for the small element. Closed loops are opened to form a tree topology by cutting nodes. A solution algorithm is developed to implement the proposed method. Nonlinear static analyses are performed for truss structures undergoing large deformations. To demonstrate the efficiency and validity of the proposed method, one numerical example is solved. A conventional linear element stiffness matrix is used to form the equations of equilibrium. The analysis results show that the proposed method has a better convergence behavior than the conventional method.

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