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변화폭 원호형 띠기초의 휨-비틀림 자유진동

Flexural-Torsional Free Vibrations of Circular Strip Foundation with Variable Breadth

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Key Words : Flexural-torsional free vibration(휨-비틀림 자유진동), circular strip(원호형 띠기초), variable width(변화폭), rotatory inertia(회전관성), torsional inertia(비틀림관성).

ABSTRACT

This paper deals with the flexural-torsional free vibrations of circular strip foundations with a variable breadth. The breadth of strip foundation varies with the linear functional fashion, which is symmetric about the mid-arc. Differential equations governing free vibrations of such foundations are derived, in which Winkler foundation is considered as the model of elastic soils. Effects of the rotatory and torsional inertias and shear deformation are included in the governing equations. Differential equations are numerically solved to calculate the natural frequencies. In the numerical examples, the free-free end constraint is considered. Effects of the rotatory and torsional inertias and shear parameter on the natural frequencies are reported. Parametric studies between frequency parameters and various system parameters are investigated.

1. Introduction

Since soil-structure interactions are one of the most important structural subjects in the foundation engineering, much study concerning the soil-structure interactions had been carried out. Structures related to the soil-structure interactions should be modeled as structures resting on or embedding in the elastic soils. One of typical structures related to the soil-structure interactions is the strip foundation which is basically defined as the beam or strip rested on or supported by elastic soil.

During the past few decades, dynamic studies on the strip foundations have been frequently investigated by many researchers. References and their citations include the governing equations and the significant historical literature on the free vibrations of beams resting on elastic foundations i.e. the strip foundations.[1,2] However, the most objective structures in such studies were the uniform members even though the real structural systems consist of many non-uniform members. Actually, non-uniform members as well as uniform ones are often erected in civil engineering works. Such typical structures include the circular strip foundations which support various loadings like the buildings, storages, and mechanical machines.

From these viewpoints, this paper aims to theoretically investigate dynamics of the circular strip foundations and also to present practically engineering data for the design of strip foundations. This paper deals with the free vibration analysis of strips which have solid rectangular cross-sections with variable breadth and constant depth, i.e. circular tapered strip foundations. In this study, the soils/elastic foundations which support the strips are modeled as a Winkler foundation and the variable breadth of the strips is assumed to be varied in the linear functional fashion. Differential equations governing the free, out-of-plane vibrations of such circular strip foundations are derived, in which effects of the rotatory and torsional inertias and also shear deformation are included although the warping of the cross-section is excluded. Boundary conditions for the free end are also derived.

Governing differential equations are numerically solved for obtaining the natural frequencies. In the numerical examples, the free-free end constraint is considered. Effects of the rotatory and torsional inertias on the natural frequencies are reported.

2. Circular strip with variable breadth

Figure 1 shows the horizontally curved circular strip with the solid rectangular cross-section and its dimensions. The radius and subtended angle are depicted as ρ and α , respectively. The typical point along the strip is defined by the polar coordinates (ρ,θ) in which θ is measured from the radius of left end. The strip with $\alpha = 2\pi$ becomes a complete ring structure. However, it is rational to apply the values of α in the region less than 2π because expansion joints are periodically installed in the real strip foundations. As shown in this figure, the depth H of the rectangular cross-section is constant along the coordinate θ, while the breadth B varies with θ. Breadths of both far ends ($\theta = 0$ and $\theta = \alpha$) are B_a and the breadth at mid-arc $(\theta = \alpha / 2)$ is B_c.

For defining the variable breadth B , the section ratio m and depth ratio n are introduced as follows.

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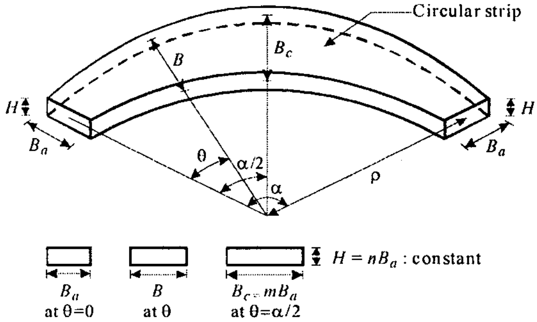

Figure 1. Circular strip having rectangular cross-section with variable breadth.

$$m = B_c / B_a, \quad n = H / B_a. \tag{1,2}$$

It is natural that the variation of breadth should be arbitrary. In this study, the breadth B is varied in a linear functional fashion with the coordinate θ and is symmetrical about the mid-arc. The equation of B which is a function of θ can now be expressed as follows.

$$B = B_a(c_1 + c_2\theta), 0 \le \theta \le \alpha, \tag{3}$$

where the coefficients c_1 and c_2 for $0 \le \theta \le \alpha/2$ are

$$c_1 = 1, \quad c_2 = (2/\alpha)(m-1), \tag{4.1,4.2}$$

and the coefficients c_1 and c_2 for $\alpha/2 \le \theta \le \alpha$ are

$$c_1 = 2m - 1, \quad c_2 = (2/\alpha)(1-m). \tag{4.3,4.4}$$

Using Eqs. (1)-(4) gives cross-sectional properties of the area, second moment of inertia, polar moment of inertia and torsional constant of the rectangular cross-section, which will be used for deriving differential equations later. The results are

$$A = BH = A_a(c_1 + c_2\theta), \tag{5}$$

$$I = BH^3/12 = I_a(c_1 + c_2\theta), \tag{6}$$

$$I_p = (BH^3 + B^3H)/12$$
$$= I_a(c_1 + c_2\theta)\{1 + n^{-2}(c_1 + c_2\theta)^2\}, \tag{7}$$

$$J = C_t BH^3 = d_1 I_a\{d_2 + d_3(c_1 + c_2\theta)\}, \tag{8}$$

where $A_a = nB_a^2$ and $I_a = n^3B_a^4/12$ are the area and second moment of inertia of the cross-section of the two extreme ends, respectively. In equation (8), the numerical factor C_t is given as $C_t = (1/3)(1 - 0.63H/B), H/B \le 1$ for the rectangular cross-section [20]. Using this numerical factor, the coefficients d_1, d_2 and d_3 in equation (8) are determined subsequently as follows.

$$d_1 = 4(1 - 0.63n), \quad d_2 = 1/(1 - 1.587/n), \tag{9.1,9.2}$$

$$d_3 = 1/(1 - 0.63n) \tag{9.3}$$

It is noted that equations (9.1)-(9.3) are valid in the ranges of $n \le 1$ and $n/m \le 1$ in order that $H/B \le 1$.

It is recalled that any other functions of the variable breadths instead of linear one chosen in this study should be selected and then, the corresponding cross-sectional properties can be similarly obtained.

3. Mathematical model

Shown in Fig. 2 is the circular strip foundation, i.e. circular strip on an elastic soil foundation, whose cross-sectional properties are already defined. Each end is either clamped or hinged or free. The dashed line is the un-deformed shape in the static state, while the solid line is one of typical deformed shapes caused by free vibrations, which is called as mode shape. Deformation variables of the vertical deflection, rotation due to pure bending, shear distortion and twist angle are denoted by v, ψ, β and ϕ, respectively. Depicted by R_v and R_T due to the foundation whose modulus is K are the vertical and torsional reactions.

In this paper, the foundations are assumed to follow the hypothesis proposed by Winkler. Figure 3 shows the restoring reactions of R_v and R_T due to v and ϕ, respectively, at any coordinate θ. In this figure, B, already defined in Fig. 1, is the breadth of the cross-section. The vertical reaction R_v is caused by the vertical deflection v. The discrepancy of v between out-and in-side extremes of the strip element is obviously caused by ϕ under the assumption that there is no bend along the radial direction.[8] As the result of this discrepancy, the Winkler foundation has the torsional reaction R_T. It is evident that the pressure of contact surface between strip and Winkler foundation is varied with r in a linear fashion. Here, r is the coordinate in the radial direction with the origin at the centroid of the rectangular cross-section depicted as o' in Fig. 3.

The relation between the pressure and deflection of the foundation surface at r can be expressed in the form

$$p(r,\theta) = Kz(r,\theta), \quad -B/2 \le r \le +B/2, \tag{10}$$

where $p(r,\theta)$ and $z(r,\theta)$ are the pressure and deflection of the contact surface, respectively, and K is the foundation modulus. From Fig. 3, one can find that

$$z(r,\theta) = v - \phi r, \quad -B/2 \le r \le +B/2. \tag{11}$$

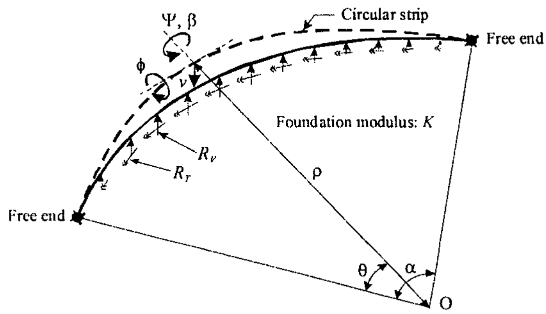

Figure 2. Circular strip foundation and its variables

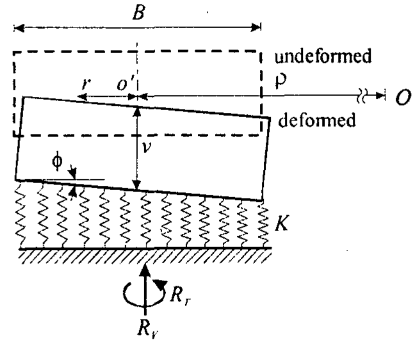

Figure 3. Reactions of foundation

Hence, both reactions R_v and R_T at coordinate θ can now be calculated by using Eqs. (10) and (11) as follows.

$$R_v = KBv, \quad R_T = (KB^3/12)\phi . \tag{12,13}$$

The three equations for "dynamic equilibrium" of the element are

$$\rho^{-1}Q' - F_v - R_v = 0, \tag{14}$$

$$\rho^{-1}M' - Q + \rho^{-1}T + C_\psi = 0, \tag{15}$$

$$\rho^{-1}M - \rho^{-1}T' + C_\phi + R_T = 0. \tag{16}$$

To facilitate the numerical studies and to obtain the most general results for this class of problem, the following non-dimensional system parameters are defined.

$$\eta = v/\rho, \quad b = B_a/\rho, \tag{17,18}$$

$$k = KB_a\rho^4/(\pi^4 EI_a), \quad g = G/E \tag{19,20}$$

where η is the normalized deflection, b is the contact ratio between strip and foundation, k is the foundation parameter and g is the shear parameter. And the frequency parameter is defined by

$$C_i = \omega_i \rho^2 \sqrt{\gamma A_a/(EI_a)} \tag{21}$$

which is expressed in terms of the i th frequency $\omega = \omega_i$, $i = 1,2,3,4,5\cdots$ where i is the mode number.

The differential equations governing free, out-of-plane vibrations of the strip foundation are derived by using all the equations mentioned above. First, the cross-sectional properties are substituted into the stress resultants, inertia loadings and foundation reactions, respectively. Second, the first derivatives of Q', M' and T' are obtained. Third, the stress resultants with the corresponding derivatives, inertia loadings and foundation reactions are then substituted into the three equations of dynamic equilibrium. Finally, non-dimensional system parameters are used. The results are

$$\eta'' = a_1\eta' + (b_1 + b_2C_i^2)\eta + \psi' - a_1\psi, \tag{22}$$

$$\psi'' = b_3\eta' + a_1\psi' + (a_2 - b_3 + b_4C_i^2)\psi + (a_2+1)\phi' - a_1\phi, \tag{23}$$

$$\phi'' = (a_3 - 1)\psi' + a_4\psi + a_4\phi' + (-a_3 + a_5 + a_6C_i^2)\phi, \tag{24}$$

where the coefficients in Eqs. (22)-(24) are as follows.

$$a_1 = -c_2/(c_1 + c_2\theta), \tag{25.1}$$

$$a_2 = d_1g\{d_2/(c_1 + c_2\theta) + d_3\}, \tag{25.2}$$

$$a_3 = -1/[d_1g\{d_2/(c_1 + c_2\theta) + d_3\}], \tag{25.3}$$

$$a_4 = -c_2d_3/\{d_2 + d_3(c_1 + c_2\theta)\}, \tag{25.4}$$

$$a_5 = \pi^4 b^2 k(c_1 + c_2\theta)^3\{d_2 + d_3(c_1 + c_2\theta)\}/(12d_1g), \tag{25.5}$$

$$a_6 = -b^2\{n^2 + (c_1 + c_2\theta)^2\} \times [12d_1g\{d_2/(c_1 + c_2\theta) + d_3\}]^{-1} \tag{25.6}$$

and

$$b_1 = \pi^4 n^2 b^2 k/(12fg), \tag{25.7}$$

$$b_2 = -n^2b^2/(12fg), \tag{25.8}$$

$$b_3 = -12fg/(n^2b^2), \tag{25.9}$$

$$b_4 = -n^2b^2/12 .(31.10)$$

Each end of the strip foundation is free. The boundary conditions for the free end ($\theta = 0$ or $\theta = \alpha$) are given by

$$\eta' - \psi = 0, \quad \phi - \psi' = 0, \quad \psi + \phi' = 0, \tag{26-28}$$

4. Numerical examples and discussion

The numerical methods described by Lee *et. al*[4] are used to solve the differential equations. First, the Runge-Kutta method is used to integrate the differential Eqs. (22)-(24) subjected to the boundary conditions of Eqs. (26)-(28). Second, the determinant search method combined with the Regula-Falsi method is used to determine the eigenvalue C_i of the differential equations.

Table 1 shows effects of rotatory and torsional inertias on C_i ($i = 1,2,3,4,5$) for the strip parameters: $\alpha = \pi/6$, $\pi/4$ and $\pi/3$, $m = 1.5$, $n = 0.3$, $b = 0.2$, $k = 30$ and $g = 0.42$. When the rotatory and torsional inertias

Table 1. Effects of rotatory (E_R) and torsional (E_T) inertias on frequency parameter C_i [a]

α	E_R	E_T	Frequency parameter, C_i				
			$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$\pi/6$	0	0	-	54.06	98.94	-	221.9
	1	0	-	53.72	96.56	-	212.2
	0	1	**42.05**	53.98	101.3	**118.6**	201.6
	1	1	**42.03**	53.59	99.01	**118.5**	201.5
$\pi/4$	0	0	-	54.06	65.64	-	112.3
	1	0	-	53.93	64.90	-	109.8
	0	1	**42.43**	53.96	66.52	**83.18**	113.6
	1	1	**42.42**	53.79	65.81	**83.07**	111.2
$\pi/3$	0	0	-	54.06	58.02	-	77.61
	1	0	-	54.00	57.64	-	76.54
	0	1	**42.69**	53.92	58.26	**66.47**	79.38
	1	1	**42.68**	53.83	57.90	**66.30**	78.50

[a] $m = 1.5$, $n = 0.3$, $b = 0.2$, $k = 30$ and $g = 0.42$.
[b] Bold lettered figure: torsional mode

are excluded, the coefficients b_4 and a_6 of the differential equations which are related to the rotatory and torsional inertias, respectively, are merely deleted. Both rotatory and torsional inertias always depress C_i values. It is apparent that if the torsional inertia is excluded ($E_T = 0$), the C_i values of torsional modes, bold lettered figures in this table, can not be calculated explicitly. Therefore it is very important to include the torsional inertia in the free, out-of-plane vibration analysis of circular strip foundations. Additionally, it is fact that the effects of rotatory inertia on torsional frequencies (bold lettered figures) are negligible comparing those on flexural frequencies.

Figure 4 shows relationships between C_i and k for the beam with $\alpha = \pi/3$, $m = 1.5$, $n = 0.3$, $b = 0.2$ and $g = 0.42$. The C_i values increase as the value of k is increased. Each mode of natural frequencies is either symmetric or antisymmetric since geometry of the strip foundation is symmetric. In this figure, the symmetric and antisymmetric modes are depicted as 'S' and 'A', respectively. Also, each mode is either flexural or torsional as shown in Table 1. The flexural mode is depicted as 'F' and torsional mode as 'T'. In this figure, two mode shapes exist at a single frequency parameter where two frequency curves meet. The third and fourth modes have the same frequency parameters $C_3 = C_4 = 75.4$ for $k = 54.9$ (marked ■). Therefore, the third and fourth modes are changed from symmetric and flexural to antisymmetric and torsional at $k = 54.9$. It is natural that the value of C_1 $(i = 1)$ for $k = 0$, namely without foundation, since the first mode is rigid one.

Figure 5 shows C_i versus n curves for the beam with $\alpha = \pi/3$, $m = 1.5$, $b = 0.2$, $k = 30$ and $g = 0.42$. The C_i values decrease as the value of n is increased. However, the first frequencies rather increase than decrease for the higher value of n more than about $n = 0.7$.

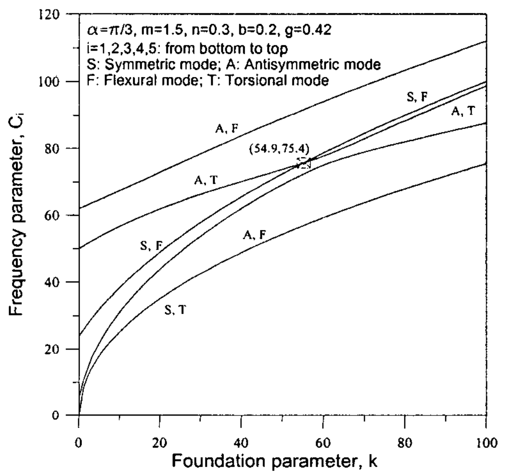

Figure 5. C_i versus k curves

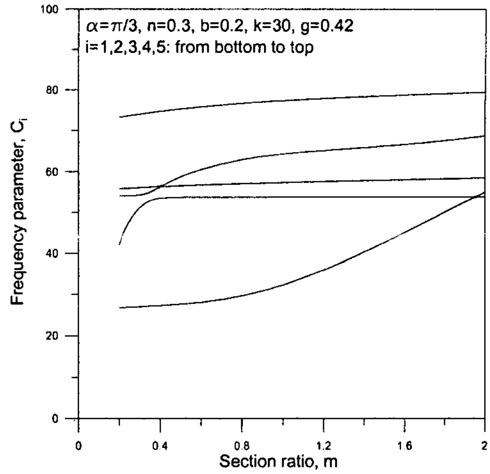

Figure 6. C_i versus m curves

5. Concluding remarks

Differential equations governing free vibrations of circular strip foundations with the variable breadth are derived, in which the elastic soils are modeled as Winkler foundation. The effects of the rotatory and torsional inertias and shear deformation are included. Differential equations are solved numerically for calculating the frequency parameters. In the numerical examples, the five lowest frequency parameters were calculated. The rotatory and torsional inertias always depress the frequency parameters. If the torsional inertia is excluded, frequency parameters of the torsional modes can not be obtained explicitly. It is expected that the results of this study can be used in designing circular strip foundations especially when subjected to dynamic loads.

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