Designs for Factorial Experiment

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Abstract

Factorial experiments are studied in this paper. The Designs, thus, have factorial balance with respect to estimable main effects and interactions. John and Lewis (1983) considered generalized cyclic row-column designs for factorial experiments. A simple method of constructing confounded designs using the classical method of confounding for block designs is described in this paper.

1. Introduction

Block designs for factorial experiments have been extensively investigated in the literature over the last several decades. In some experiments, however, two sources of extraneous variability must be controlled where Latin square and Youden square designs are quite useful. A more flexible class of two-dimensional designs is provided by row-column designs as Latin squares and squares do not always suit the requirements of the experimenter. Yates (1937), and Rao (1946) gave some row-column designs for 2-level and 3-level factorial experiments. The algorithmic methods of Patterson (1976), Bailey et al.

(1977), and Patterson and Bailey (1978) are quite general in that they are capable of producing designs with various blocking structures, including row-column designs. John and Lewis (1983) developed
the generalized cyclic method of construction for obtaining row-column designs. They also derived main effect and interaction efficiency factors for generalized cyclic designs, and gave some guidelines for choosing the row and column component designs appropriately.

It is clear that the developments in the area of row-column designs for factorial designs are relatively few. The purpose of this paper, therefore, is to present a simple method of obtaining confounded row-column designs based on the classical method of confounding. Thus, the method is applicable to symmetrical factorial experiments, with number of levels a prime or a prime power. The model and some preliminaries of row-column designs are presented in Section 2. The method of construction is presented in Section 3. The method is illustrated with the help of several examples.

2. Model and Preliminaries

Consider a row-column design D having v treatments, p rows and q columns. Let r denote the constant number of treatment replications The model for a row-column design is given by,

$$Y_{ijk} = \mu + \tau_i + \rho_j + \gamma_k + \epsilon_{ijk}$$

where Y_{ijk} is the observation from the jth row, kth column to which the ith treatments has been applied, μ is the overall mean, τ_i, ρ_j, γ_k are the effects of the ith treatment, jth row, and kth column respectively, and $\epsilon' s$ are uncorrelated random errors with mean zero and variance σ^2.

Let N_1 and N_2 denote the treatment-row and treatment-column incidence matrices respectively. The component designs corresponding to N_1 an N_2 will be denoted by D_1 and D_2 respectively. The intra-block reduced normal equations for estimating the vector of treatment parameters $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_v)', \tau_i$ being the i th treatment effect, are given by $C\boldsymbol{\tau} = \boldsymbol{Q}$ with

$$C = rI - \frac{1}{q}N_1N_1' - \frac{1}{p}N_2N_2' + \frac{r^2}{pq}\mathbf{1}\mathbf{1}',$$

where I denotes the identity matrix and 1 denotes the column vector of 1's of size v each. It is easy to verify that

$$C = C_1 + C_2 - C_b \tag{2.1}$$

where

$$C_1 = rI - \frac{1}{q}N_1N_1'$$

$$C_2 = rI - \frac{1}{p}N_2N_2'$$

$$C_b = r(I - \frac{1}{v}\mathbf{1}\mathbf{1}')$$

are the intra-block information matrices of the component designs D_1, D_2 and D_b respectively.

Here D_b denotes a design having one block obtained by ignoring the row and column classifications, with the incidence matrix $N_b = r\mathbf{1}$.

Lemma 2.1. Suppose $\boldsymbol{u} = (u_1, u_2, \cdots, u_v)'$ with $\boldsymbol{u}'\mathbf{1} = 0$ is an eigen vector of both $C_1$and C_2.

(a) If $\boldsymbol{u}'\boldsymbol{\tau}$ is confounded in the component design $D_1(D_2)$ and it is not confounded in the component deign $D_2(D_1)$, then it is confounded in the row-column design D.

(b) If $\boldsymbol{u}'\boldsymbol{\tau}$ is unconfounded in both the component designs D_1 and D_2, then it is unconfounded in the row-column design D as well.

Proof. (a) clearly $\boldsymbol{u}'\boldsymbol{\tau}$ is unconfounded in D_b. Suppose it is confounded in the component design D_1 only. Then, using $C_1\boldsymbol{u}=\mathbf{0}$, $C_2\boldsymbol{u}=C_b\boldsymbol{u}=r\boldsymbol{u}$ in equation (2.1) we get $C\boldsymbol{u}=\mathbf{0}$.

(b) Now suppose $\boldsymbol{u}'\boldsymbol{\tau}$ is unconfounded in both D_1 and D_2. Then , $C_1\boldsymbol{u}=C_2\boldsymbol{u}=C_b\boldsymbol{u}=r\boldsymbol{u}$. Thus, $C\boldsymbol{u}=r\boldsymbol{u}$. Hence the lemma.

3. The Method of Construction

As mentioned in the introduction section, attention will be restricted to equireplicate s^m factorial experiments involving m factors $F_1, F_2, \cdots, F_m$, having s levels each. The treatment combinations will be denoted by m-tuples $a_1a_2\cdots a_m$. Where the s levels of a factor are coded as $0,1,\cdots,s-1$ with $a_i\epsilon\{0,1,\cdots,s-1\}, i=1,2,\cdots,m$.Consider the row-column design D for a s^m factorial experiment. Let $p=s^{m_1}$, $q=s^{m_2}$ with $pq=s^{m_1+m_2}=rs^m$.
Thus,

$$r=s^{m_1+m_2-m}.$$

Since $r\geq 1$, we have,

$$m_1+m_2\geq m.$$

Let $A_1, A_2, \cdots, A_{m-m_2}$ denote the independent interactions confounded between $p=s^{m_1}$ rows of D. Then, a total of $s^{m-m_2}-1$ treatment degrees freedom are confounded between $p=s^{m_1}=rs^{m-m_2}$ rows of D. The total number of effects confounded between rows, each effect having $s-1$ degrees of freedom, are then given by $(s^{m-m_2}-1)/(s-1)$.. Thus, the number of generalized interactions confounded between rows of D is given by

$$g_1 = \frac{s^{m-m_2}-1}{s-1} - (m-m_2).$$

Let these generalized interactions be denoted by $A_{m-m_2+1}, A_{m-m_2+2}, \cdots,$ $A_{m-m_2+g_1}$.

Now let $B_1, B_2, \cdots, B_{m-m_1}$ denote the independent interactions confounded between $q = s^{m_2} = rs^{m_2}$ columns of D. Then, following a similar argument as above, the number generalized interactions confounded with rs^{m_2} columns of D is given by

$$g_2 = \frac{s^{m-m_1}-1}{s-1} - (m-m_1).$$

These g_2 generalized interactions will be denoted by $B_{m-m_1+1}, B_{m-m_1+2},$ $\cdots, B_{m-m_1+g_2}$.

The factorial effects $A_1, A_2, \cdots, A_{m-m_2+g_1}, B_1, B_2, \cdots, B_{m-m_1+g_2}$ are to be chosen such that they are all distinct.

We first construct the key or the principal block for rows by confounding $m-m_2$ independent interactions $A_1, A_2, \cdots, A_{m-m_2}$ between rows of D. Let this row key block be denoted by $(a_1^i a_2^i \cdots a_m^i, i = 1, 2, \cdots, s^{m_2})$ where $a_\ell^i \epsilon \{0, 1, \cdots, s-1\}$ with $a_\ell^1 = 0, \ell = 1, 2, \cdots, m$. Next the column key block is similarly obtained by confounding $m-m_1$ independent interactions $B_1, B_2, \cdots, B_{m-m_1}$ between columns of D. Let the column key block be denoted by $(b_1^j\, b_2^j \cdots b_m^j, j = 1, 2, \cdots, s^{m_1})$, where $b_\ell^j \epsilon \{0, 1, \cdots, s-1\}$ with $b_\ell^1 = 0, \ell = 1, 2, \cdots, m$. Then the treatment combination in the ℓ_1^{th} row and ℓ_2^{th} column of D is given by $c_1, c_2, \cdots, c_m$, with $c_\ell = a_\ell^{\ell_2} + b_\ell^{\ell_1}$, $\ell = 1, 2, \cdots, m$, where the addition is done mod(s).

Example 2.1 A 2^4 experiment using a row-column design with $p=q=2^2$. Thus $s=2, m=4, m_1=m_2=2, m-m_1=m-m_2=2$. Let $A_1=F_1F_2, A_2=F_3F_4$, $B_1=F_1F_2F_3,\ B_2=F_2F_3F_4$. Then $A_3=F_1F_2F_3F_4,\ B_3=F_1F_4$. The row and column key blocks and the resulting row-column design are given below.

Column Key Block	Row Key Block 0000 1100 0011 1111
0000	0000 1100 0011 1111
0110	0110 1010 0101 1001
1101	1101 0001 1110 0010
1011	1011 0111 1000 0100

The above single replicate design completely confounds the interactions $F_1F_2, F_3F_4, F_1F_2F_3F_4, F_1F_2F_3, F_2F_3F_4$ and F_1F_2.. All the main other interactions are estimated with full efficiency. A design which completely confounds only the four-factor interaction $F_1F_2F_3F_4$ can be constructed by adding a second replicate, obtained using $A_1=F_1F_3, A_2=F_2F_4$, $B_1=F_1F_3F_4, B_2=F_1F_2F_4$.

Example 2.2 A 2^4 experiment using a row-column design with $p=2^2,\ q=2^3$. Here $s=2, m=4,\ m_1=2, m_2=3,\ \ m-m_1=2,\ \ m-m_2=1$.

Let $A_1=F_1F_2F_3F_4,\ B_1=F_1F_2F_3,\ B_2=F_2F_3F_4$. Then, $B_3=F_1F_4$. The row and column key blocks and the resulting row-column design are given below.

Column Key Block	Row Key Block							
	0000	0011	0101	0110	1100	1010	1001	1111
0000	0000	0011	0101	0110	1100	1010	1001	1111
0110	0110	0101	0011	0000	1010	1100	1111	1001
1101	1101	1110	1000	1011	0001	0111	0100	0010
1011	1011	1000	1110	1101	0111	0001	0010	0100

Note that each treatment is replicated $r = s^{m_1+m_2-m} = 2$ times in the above design. The interactions F_1F_4, $F_1F_2F_3$, $F_2F_3F_4$ and $F_1F_2F_3F_4$. are completely confounded, and there is no loss of information on any of the main effects and other interactions.

Example 2.3 A 3^3 experiment, with $p = 3, q = 3^2$.

Here $s = 3, m = 3, m_1 = 1, m_2 = 2$, $m - m_1 = 2, m - m_2 = 1$. Let $A_1 = F_1F_2F_3$, $B_1 = F_1F_2F_3^2$, $B_2 = F_2F_3$. Then $B_3 = F_1F_2^2$, $B_4 = F_1F_3$. The design is then as follows.

Column Key Block	Row Key Block								
	000	102	012	201	021	111	120	210	222
000	000	102	012	201	021	111	120	210	222
112	112	211	121	010	100	220	202	022	001
221	221	020	200	122	212	002	011	101	110

The above single replicate design completely confounds the interactions $F_1F_2F_3$, $F_1F_2F_3^2$, F_2F_3, $F_1F_2^2$, F_1F_3, having degrees of freedom each. All other factorial effects are estimated with full efficiency.

References

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(Example 2.1) 2^4 experiment with $p=q=2^2$.

1) Note

(1) # of level $s=2$: $(0,1)$

(2) # of factor $m=4$: $F_1F_2F_3F_4$

(3) • The number of independent interactions confounded between $p=2^2$ rows of D is $m-m_2=2$

• The number of generalized interactions confounded between $p=2^2$ rows of D is

$$g_1=\frac{s^{m-m_2}-1}{s-1}-(m-m_2)=1$$

(4) • The number of independent interactions confounded between $q=2^2$ columns of D is $m-m_1=2$

• The number of generalized interactions confounded between $q=2^2$ columns of D is

$$g_2=\frac{s^{m-m_1}-1}{s-1}-(m-m_1)=1$$

(5) The number of repeats is $r=s^{m_1+m_2-m}=1$

2) We construct the row component design by confounding two independent interactions $A_1=F_1F_2$ and $A_2=F_3F_4$ (Note : The generalized interactions is $A_3=F_1F_2F_3F_4$)

(1) The defining contrasts for A_1 is $L_1=x_1+x_2=0\ or\ 1$

The defining contrasts for A_2 is $L_2=x_3+x_4=0\ or\ 1$

We get 4 blocks as follows :

Row key block		2^{nd} block		3^{rd} block		4^{th} block	
$x_1+x_2=0$ $x_3+x_4=0$		$x_1+x_2=0$ $x_3+x_4=1$		$x_1+x_2=1$ $x_3+x_4=0$		$x_1+x_2=1$ $x_3+x_4=1$	
	code		code		code		code
0 0 0 0	1	0 0 0 1	2	0 1 0 0	5	0 1 0 1	6
0 0 1 1	4	0 0 1 0	3	0 1 1 1	8	0 1 1 0	7
1 1 0 0	13	1 1 0 1	14	1 0 0 0	9	1 0 0 1	10
1 1 1 1	16	1 1 1 0	15	1 0 1 1	12	1 0 1 0	11

Since $r=1$, the row component design has 4blocks.

(2) Incidence matrix

$$N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(3) Information matrix

$$C_1 = rI_{16} - \frac{1}{q}N_1N_1' = I_{16} - \frac{1}{4}N_1N_1'$$

$$= \frac{1}{4}\begin{pmatrix}
3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
-1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3
\end{pmatrix}$$

3) We construct the column component design by confounding two independent interactions $B_1 = F_1F_2F_3$ and $B_2 = F_2F_3F_4$. Also, the generalized interaction is $B_3 = F_1F_2F_3F_4$.

(1) The defining contrast for B_1 is $L_3 = x_1 + x_2 + x_3 = 0$ or 1

The defining contrast for B_2 is $L_4 = x_2 + x_3 + x_4 = 0$ or 1

We get 4 blocks as follow :

Column key block		2^{nd} block		3^{rd} block		4^{th} block	
$x_1+x_2+x_3=0$ $x_2+x_3+x_4=0$		$x_1+x_2+x_3=0$ $x_2+x_3+x_4=1$		$x_1+x_2+x_3=1$ $x_2+x_3+x_4=0$		$x_1+x_2+x_3=1$ $x_2+x_3+x_4=1$	
	code		code		code		code
0 0 0 0	1	0 0 0 1	2	0 0 1 1	4	0 0 1 0	3
0 1 1 0	7	0 1 1 1	8	0 1 0 1	6	0 1 0 0	5
1 0 1 1	12	1 0 1 0	11	1 0 0 0	9	1 0 0 1	10
1 1 0 1	14	1 1 0 0	13	1 1 1 0	15	1 1 1 1	16

(2) Incidence matrix :

Since $r = 1$, the incidence matrix N_2 for column component design has 16 rows for the 16 treatments and 4 columns for the 4 blocks.

$$N_2 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

(3) Information matrix

$$C_2 = rI_{16} - \frac{1}{p} N_2 N_2{}' = I_{16} - \frac{1}{4} N_2 N_2{}'$$

$$= \frac{1}{4}\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 3 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 3 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 3 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}$$

4) One block design :

(1) Incidence matrix

$$N_b = r1_{16} = 1_{16}$$

(2) Information matrix

$$C_b = rI_{16} - \frac{r}{v} 1_{16} 1_{16}{}' = I_{16} - \frac{1}{16} J_{16}$$

$$= \frac{1}{16}\begin{pmatrix}
15 & & & \\
 & 15 & -1 & \\
 & & \ddots & \\
 & -1 & & \ddots \\
 & & & & 15
\end{pmatrix}_{16\times 16}$$

5) Row column design

(1) Information matrix

$C = C_1 + C_2 - C_b$

$$
= \frac{1}{16}\begin{pmatrix}
9 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & 1 & -3 \\
1 & 9 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & -3 & -3 & -3 & 1 \\
1 & -3 & 9 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & -3 & -3 \\
-3 & 1 & 1 & 9 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & -3 & -3 \\
1 & 1 & -3 & 1 & 9 & 1 & 1 & -3 & -3 & -3 & 1 & -3 & 1 & 1 & 1 & -3 \\
1 & 1 & 1 & -3 & 1 & 9 & -3 & 1 & -3 & -3 & -3 & 1 & 1 & 1 & -3 & 1 \\
-3 & 1 & 1 & 1 & 1 & -3 & 9 & 1 & 1 & -3 & -3 & -3 & 1 & -3 & 1 & 1 \\
1 & -3 & 1 & 1 & -3 & 1 & 1 & 9 & -3 & 1 & -3 & -3 & -3 & 1 & 1 & 1 \\
1 & 1 & 1 & -3 & -3 & -3 & 1 & -3 & 9 & 1 & 1 & -3 & 1 & 1 & -3 & 1 \\
1 & 1 & -3 & 1 & -3 & -3 & -3 & 1 & 1 & 9 & -3 & 1 & 1 & 1 & 1 & -3 \\
1 & -3 & 1 & 1 & 1 & -3 & -3 & -3 & 1 & -3 & 9 & 1 & -3 & 1 & 1 & 1 \\
-3 & 1 & 1 & 1 & -3 & 1 & -3 & -3 & -3 & 1 & 1 & 9 & 1 & -3 & 1 & 1 \\
-3 & -3 & 1 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 9 & 1 & -3 & 1 \\
-3 & -3 & -3 & 1 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 9 & -3 & 1 \\
1 & -3 & -3 & -3 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 1 & 1 & -3 & 9 & 1 \\
-3 & 1 & -3 & -3 & -3 & 1 & 1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & 9
\end{pmatrix}
$$

6) Remark

Obtained row-column design can be rearrange as follows

(1) Defining contrast :

① $A_1 = F_1F_2 \;\Rightarrow\; L_1 = x_1 + x_2 = 0 \; or \; 1$

$A_2 = F_3F_4 \;\Rightarrow\; L_2 = x_3 + x_4 = 0 \; or \; 1$

② $B_1 = F_1F_2F_3 \;\Rightarrow\; L_3 = x_1 + x_2 + x_3 = 0 \; or \; 1$

$B_2 = F_2F_3F_4 \;\Rightarrow\; L_4 = x_2 + x_3 + x_4 = 0 \; or \; 1$

(2) Key block

① Row :

$$
\left.\begin{array}{l} L_1 = x_1 + x_2 = 0 \\ \\ L_2 = x_3 + x_4 = 0 \end{array}\right\} \;\Rightarrow\; \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}
$$

② Column :

$$\left.\begin{aligned} L_3 = x_1 + x_2 + x_3 = 0 \\ L_4 = x_2 + x_3 + x_4 = 0 \end{aligned}\right\} \Rightarrow \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{matrix}$$

(3) Treatment Combination Code :

Treatment Combination	0000	0001	0010	0011	0100	0101	0110	0111
code	1	2	3	4	5	6	7	8

Treatment Combination	1000	1001	1010	1011	1100	1101	1110	1111
code	9	10	11	12	13	14	15	16

(4) Row column design :

Column Key Block	Row Key Block							
	0 0 0 0		0 0 1 1		1 1 0 0		1 1 1 1	
		code		code		code		code
0 0 0 0	0 0 0 0	1	0 0 1 1	4	1 1 0 0	13	1 1 1 1	16
0 1 1 0	0 1 1 0	7	0 1 0 1	6	1 0 1 0	11	1 0 0 1	10
1 0 1 1	1 0 1 1	12	1 0 0 0	9	0 1 1 1	8	0 1 0 0	5
1 1 0 1	1 1 0 1	14	1 1 1 0	15	0 0 0 1	2	0 0 1 0	3

(5) Incidence matrix

① Row :

by the row-column treatment combination , i^{th} row is i^{th} block $(i = 1, 2, 3, 4)$ with coded number.

② Column :

by the row-column treatment combination , i^{th} column is i^{th} block $(i = 1, 2, 3, 4)$ with coded number.

(6) Information matrix :

① $C_1 = rI_{16} - \frac{1}{q} N_1 N_1'$

② $C_2 = rI_{16} - \frac{1}{p} N_2 N_2'$

③ $C_b = rI_{16} - \frac{r}{v} 1_{16} 1_{16}' = rI_{16} - \frac{r}{v} J_{16}$

④ $C = C_1 + C_2 - C_b$