Estimation for the Weibull Distribution Based on Censored Samples

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Abstract

We consider the problem of estimating the scale and shape parameter of the Weibull distribution based on censored samples. we propose the approximate maximum likelihood estimators (AMLEs) of the scale and shape parameters in the Weibull distribution based on Type-II censored samples. We compare the proposed estimators in the sense of mean squared error (MSE) .

Keywords : Approximate maximum likelihood estimator, Extreme value distribution, Type-II censored sample, Weibull distribution.

1. Introduction

The probability density function (pdf) of the random variable X having the Weibull distribution is given by

$$f(x)=\frac{\delta}{\Theta^{\delta}}x^{\delta-1}\exp\left\{-\left(\frac{x}{\Theta}\right)^{\delta}\right\},\quad x>0,\ \Theta>0,\ \delta>0. \tag{1.1}$$

The estimation of the parameters in a censored sample, has been investigated by many authors such as Balakrishnan (1989), Balakrishnan and Cohen (1991), and Fei and Kong (1995).

Balakrishnan et al. (1995) obtained the AMLEs of the location and scale parameters of the extreme value distribution based on multiply Type-II censored samples

Kang (1996, 2003) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and the AMLEs of the location and scale parameters of the exponential distribution based on Multiple Type-II censored samples.

Recently, Kang et al. (2004) introduced the AMLE of the scale parameter of the Weibull distribution based on multiply Type-II censored samples.

In this paper, we propose the AMLEs of the scale and shape parameters in the Weibull distribution based on Type-II censored samples by using the relationship of Weibull and extreme value distributions. We compare the proposed estimators in

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the sense of MSE.

2. Estimation of the parameters

Consider the Weibull distribution with the pdf (1.1) and the cumulative distribution function (cdf)

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\theta}\right)^{\delta}\right\}, \quad x > 0. \tag{2.1}$$

Let

$$X_{r+1:n} \le X_{r+2:n} \le \cdots \le X_{n-s:n} \tag{2.2}$$

be a doubly Type-II censored sample from the Weibull distribution with pdf (1.1), where the first r and the last s observations are censored.

The estimation for the shape parameter of the Weibull distribution based on Type-II censored samples is difficult. So we will use the transformation which changes the Weibull distribution into the extreme value distribution.

Let X be a random variable with pdf (1.1), then the pdf of $Y = \ln X$ is

$$g(y) = \frac{1}{\sigma} e^{\frac{y-\mu}{\sigma}} \exp\left(-e^{\frac{y-\mu}{\sigma}}\right), \quad -\infty < y < \infty, \tag{2.3}$$

where $\sigma = \frac{1}{\delta}$, $\mu = \ln\theta$ and the cdf of Y is

$$G(y) = 1 - \exp\left(-e^{\frac{y-\mu}{\sigma}}\right), \quad -\infty < y < \infty. \tag{2.4}$$

That is, Y has the extreme value distribution with the location parameter μ and the scale parameter σ.

The likelihood function based on the doubly Type-II censored sample (2.2) can be written as

$$L = \frac{n!}{r!\,s!} \{G(Y_{r+1:n})\}^r \{1 - G(Y_{n-s:n})\}^s \prod_{i=r+1}^{n-s} g(Y_{i:n}). \tag{2.5}$$

By putting $Z_{j:n} = \dfrac{Y_{j:n} - \mu}{\sigma}$, the likelihood function (2.5) can be rewritten as

$$L = \frac{n!}{r!\,s!} \sigma^{-(n-r-s)} \{G(Z_{r+1:n})\}^r \{1 - G(Z_{n-s:n})\}^s \prod_{i=r+1}^{n-s} g(Z_{i:n}), \tag{2.6}$$

where $g(z) = e^z \exp(-e^z)$ and $G(z) = 1 - \exp(-e^z)$ are the pdf and cdf of the standard extreme value distribution, respectively.

Therefore, we have the log-likelihood function as follows;

$$\begin{aligned} \ln L = \; & \ln\frac{n!}{r!\,s!} + r\ln\{G(Z_{r+1:n})\} + s\ln\{1 - G(Z_{n-s:n})\} \\ & + \sum_{i=r+1}^{n-s} \ln g(Z_{i:n}) - (n-s-r)\ln\sigma. \end{aligned} \tag{2.7}$$

We want to obtain the estimators of the shape parameter when the scale parameter is known and unknown. We compute and compare the MSEs of the proposed estimators through the Monte Carlo simulations.

2.1 Estimation of the shape parameter when the scale parameter is known

First, we shall find the estimator of the scale parameter of the extreme value distribution with pdf (2.3) when the location parameter μ is known.

On differentiating the log-likelihood function (2.7) with respect to σ in turn and equating to zero, we obtain the likelihood equation as

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\left[r\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})}Z_{r+1:n} - s\frac{g(Z_{n-s:n})}{1-G(Z_{n-s:n})}Z_{n-s:n} + \sum_{i=r+1}^{n-s}\frac{g'(Z_{i:n})}{g(Z_{i:n})}Z_{i:n} + (n-s-r)\right] = 0. \tag{2.8}$$

Since the likelihood equation (2.8) is very complicated, the equation does not admit an explicit solution for σ. But if we let

$$p_i = \frac{i}{n+1}, \quad G^{-1}(p_i) = \ln(-\ln(1-p_i)) \equiv \xi_i,$$

then we may expand the following functions in Taylor series around the points ξ_{r+1}, ξ_{n-s}, and ξ_i. respectively;

$$\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})}Z_{r+1:n}, \quad \frac{g(Z_{n-s:n})}{1-G(Z_{n-s:n})}Z_{n-s:n}, \quad \frac{g'(Z_{i:n})}{g(Z_{i:n})}Z_{i:n}.$$

That is, we may approximate these functions as

$$\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})}Z_{r+1:n} \simeq \alpha_{1r} + \beta_{1r}Z_{r+1:n} \tag{2.9}$$

$$\frac{g(Z_{n-s:n})}{1-G(Z_{n-r:n})}Z_{n-r:n} \simeq -e^{\xi_{n-s}}\xi_{n-s}^2 + e^{\xi_{n-s}}(\xi_{n-s}+1)Z_{n-s:n} \tag{2.10}$$

and

$$\frac{g'(Z_{i:n})}{g(Z_{i:n})}Z_{i:n} \simeq e^{\xi_i}\xi_i^2 + [1 - e^{\xi_i} - e^{\xi_i}\xi_i]Z_{i:n}. \tag{2.11}$$

where

$$\alpha_{1r} = -\frac{\exp(\xi_{r+1})\exp(-\exp^{\xi_{r+1}})[1-\exp(\xi_{r+1}) - \exp\{-\exp(\xi_{r+1})\}]}{[\exp(-\exp^{\xi_{r+1}})]^2}\xi_{r+1}^2$$

$$\beta_{1r} = \frac{\exp(\xi_{r+1})\exp(-\exp^{\xi_{r+1}})[1-\exp(\xi_{r+1}) - \exp\{-\exp(\xi_{r+1})\}]}{[1-\exp(-\exp^{\xi_{r+1}})]^2}\xi_{r+1} + \frac{\exp(\xi_{r+1})\exp(-\exp^{\xi_{r+1}})}{1-\exp(-\exp^{\xi_{r+1}})}.$$

By substituting the equations (2.9), (2.10), and (2.11) into the equation (2.8), we obtain the approximate likelihood equation for σ as follows;

$$\frac{d\ln L}{d\sigma} \simeq \frac{1}{\sigma}\left[r(\alpha_{1r} + \beta_{1r}Z_{r+1:n}) - s\{-e^{\xi_{n-s}}\xi_{n-s}^2 + e^{\xi_{n-s}}(\xi_{n-s}+1)Z_{n-s:n}\} + \sum_{i=r+1}^{n-s}\{e^{\xi_i}\xi_i^2 + (1-e^{\xi_i} - e^{\xi_i}\xi_i)Z_{i:n}\} + (n-s-r)\right] = 0. \tag{2.12}$$

Upon solving the equation (2.12) for σ, we can derive an estimator of σ as follows;

$$\hat{\sigma}_1 = \frac{1}{r\alpha_{1r} + se^{\xi_{n-s}} + \sum_{i=r+1}^{n-s} e^{\xi_i}\xi_i^2 + (n-s-r)} \Big[r\beta_{1r}(Y_{r+1:n} - \mu) - se^{\xi_{n-s}}(\xi_{n-s}+1)(Y_{n-s:n} - \mu) + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)(Y_{i:n} - \mu)\Big]. \tag{2.13}$$

Now, we derive the single moments and the product moments of the order statistics by using the formulas of Gradshteyn and Ryzhik (1965) and Balakrishinan and Chan (1992). The results are given by

$$E(Z_{i:n}) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} \times \Big[-\frac{1}{n-j}(\gamma + \ln(n-j)) \Big], \tag{2.14}$$

$$E(Z^2_{i:n}) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^{i-j-1} \times \binom{i-1}{j} \frac{1}{n-j} \Big[\frac{\pi^2}{6} + (\gamma + \ln(n-j))^2 \Big], \tag{2.15}$$

$$E(Z_{i:n}Z_{j:n}) = \frac{n!}{(i-1)!} \sum_{l=0}^{j-i-1} (-1)^{i-j-1} \times \sum_{m=0}^{n-j} \frac{(-1)^{l+m}}{l!(j-i-1-l)!m!(n-j-m)!} \times \phi(i+l, j-i-l+\mu), \tag{2.16}$$

where the function ϕ is the double integral

$$\phi(t,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} xye^{x-te^x}e^{y-ue^y}dxdy, \quad t>0, \; u>0.$$

From the equations (2.14), (2.15), and (2.16), we can compute the expectation and the variance of the estimator $\hat{\sigma}_1$ as

$$\begin{aligned} E(\hat{\sigma}_1) &= \frac{1}{A_1}[-r\beta_{1r}E(Y_{r+1:n} - \mu) + se^{\xi_{n-s}}(\xi_{n-s}+1)E(Y_{n-s:n} - \mu) \\ &\quad - \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)E(Y_{i:n} - \mu)] \\ &= \frac{\sigma}{A_1}[-r\beta_{1r}E(Z_{r+1:n}) + se^{\xi_{n-s}}(\xi_{n-s}+1)E(Z_{n-s:n}) \\ &\quad - \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)E(Z_{i:n})], \end{aligned} \tag{2.17}$$

$$\begin{aligned} \mathrm{Var}(\hat{\sigma}_1) &= \frac{\sigma^2}{A_1^2}\Big[r^2\beta_{1r}^2\{E(Z^2_{r+1:n}) - (E(Z_{r+1:n}))^2\} \\ &\quad + (se^{\xi_{n-s}})^2(\xi_{n-s}+1)^2\{E(Z^2_{n-s:n}) - (E(Z_{n-s:n}))^2\} \\ &\quad - \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)^2\{E(Z^2_{i:n}) - (E(Z_{i:n}))^2\} \end{aligned} \tag{2.18}$$

$$-2r\beta_{1r}(se^{\xi_{n-s}})(\xi_{n-s}+1)\{E(Z_{r+1:n}Z_{i:n})-E(Z_{r+1:n})E(Z_{i:n})\}$$

$$-2r\beta_{1r}\sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)\{E(Z_{r+1:n}Z_{n-s:n})$$

$$-E(Z_{r+1:n})E(Z_{n-s:n})\}-2(se^{\xi_{n-s}})(\xi_{n-s}+1)$$

$$\times\sum_{i=r+1}^{n-s}(1-e^{\xi_i}-e^{\xi_i}\xi_i)\{E(Z_{i:n}Z_{n-s:n})-E(Z_{i:n})E(Z_{n-s:n})\}\Big],$$

where

$$A_1=-r\alpha_{1r}+se^{\xi_{n-s}}-\sum_{i=r+1}^{n-s}e^{\xi_i}\xi_i^2+(n-s-r).$$

Since $\sigma=1/\delta$, we can obtain the estimator of δ as follows;

$$\hat{\delta}_1=1/\hat{\sigma}_1. \tag{2.19}$$

It's difficult to find the expectation or variance of the estimator $\hat{\delta}_1$, so we will simulate the MSEs for the proposed estimator $\hat{\delta}_1$ of δ.

We can propose another estimator of the parameter δ by the similar method.

In the equation (2.8), we may expand the following functions in Taylor series around the points ξ_{r+1}, ξ_{n-s}, and ξ_i, respectively.

$$\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})},\quad \frac{g(Z_{n-s:n})}{1-G(Z_{n-s:n})},\quad \frac{g'(Z_{i:n})}{g(Z_{i:n})}.$$

That is

$$\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})}\simeq\alpha_{2r}+\beta_{2r}Z_{r+1:n}, \tag{2.20}$$

$$\frac{g(Z_{n-s:n})}{1-G(Z_{n-s:n})}\simeq e^{\xi_{n-s}}(1-\xi_{n-s})+e^{\xi_{n-s}}Z_{n-s:n}, \tag{2.21}$$

and

$$\frac{g'(Z_{i:n})}{g(Z_{i:n})}\simeq 1-e^{\xi_i}+e^{\xi_i}\xi_i-e^{\xi_i}Z_{i:n}, \tag{2.22}$$

where

$$\alpha_{2r}=\frac{e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})}{1-\exp(-e^{\xi_{r+1}})}-\frac{e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})[1-e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})]}{[1-\exp(-e^{\xi_{r+1}})]^2}\xi_{r+1}$$

$$\beta_{2r}=\frac{e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})[1-e^{\xi_{r+1}}\exp(-e^{\xi_{r+1}})]}{[1-\exp(-e^{\xi_{r+1}})]^2}\xi_{r+1}.$$

By substituting the equations (2.20), (2.21), and (2.22) into the equation (2.8), we obtain the approximate likelihood equation for σ as follows;

$$\frac{\partial\ln L}{\partial\sigma}\simeq-\frac{1}{\sigma}\Big[r(\alpha_{2r}+\beta_{2r}Z_{r+1:n})Z_{r+1:n}-s\{e^{\xi_{n-s}}(1-\xi_{n-s})$$

$$+e^{\xi_{n-s}}Z_{n-s:n}\}Z_{n-s:n}+\sum_{i=r+1}^{n-s}\{(1-e^{\xi_i}+e^{\xi_i}\xi_i \tag{2.23}$$

$$-e^{\xi_i}Z_{i:n})\}Z_{i:n}+(n-s-r)\Big]=0.$$

From the equation (2.23), we obtain the quadratic equation for σ as follows;

$$A_2\sigma^2 + B_2\sigma + C_2 = 0 \tag{2.24}$$

where

$$A_2 = n - r - s,$$

$$B_2 = r\alpha_{2r}(Y_{r+1:n} - \mu) - se^{\xi_{n-s}}(1 - \xi_{n-s})(Y_{n-s:n} - \mu)$$

$$+ \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} + e^{\xi_i}\xi_i)(Y_{i:n} - \mu),$$

$$C_2 = r\beta_{2r}(Y_{r+1:n} - \mu) - se^{\xi_{n-s}}(Y_{n-s:n} - \mu) - \sum_{i=r+1}^{n-s} e^{\xi_i}(Y_{i:n} - \mu).$$

On solving the equation (2.24) for σ, we derive another estimator of σ as

$$\hat{\sigma}_2 = \frac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}. \tag{2.25}$$

Therefore, we also obtain the other estimator of δ as follows;

$$\hat{\delta}_2 = 1/\hat{\sigma}_2. \tag{2.26}$$

2.2 Estimation of the shape and the scale parameters

Now, we shall obtain the estimators of the shape and scale parameters of the Weibull distribution with pdf (1.1).

On differentiating the log-likelihood function (2.7) with respect to μ, we obtain the likelihood equation as

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{\sigma}\left[r\frac{g(Z_{r+1:n})}{G(Z_{r+1:n})} - s\frac{g(Z_{n-s:n})}{1 - G(Z_{n-s:n})} + \sum_{i=r+1}^{n-s}\frac{g'(Z_{i:n})}{g(Z_{i:n})}\right] = 0. \tag{2.27}$$

The likelihood equations (2.8) and (2.27) do not admit explicit solutions for the parameters. To obtain the explicit solutions in the likelihood equations, we use the Taylor series. By substituting the equations (2.20), (2.21), and (2.22) into the equation (2.27), we obtain the approximate likelihood equation for μ as

$$\frac{\partial \ln L}{\partial \mu} \simeq r(\alpha_{2r} + \beta_{2r}Z_{r+1:n}) - s\{e^{\xi_{n-s}}(1 - \xi_{n-s}) + e^{\xi_{n-s}}Z_{n-s:n}\}$$
$$+ \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} + e^{\xi_i}\xi_i - e^{\xi_i}Z_{i:n}) = 0. \tag{2.28}$$

Upon solving the equations (2.12) and (2.28), we derive the approximate MLEs of μ and σ as

$$\hat{\mu} = \frac{D'F - DF'}{D'E - DE'} \tag{2.29}$$

and

$$\hat{\sigma} = \frac{1}{r\alpha_{1r} + se^{\xi_{n-s}} + \sum_{i=r+1}^{n-s} e^{\xi_i}\xi_i^2 + (n-s-r)} \Big[r\beta_{1r}(Y_{r+1:n} - \hat{\mu}) - se^{\xi_{n-s}}(\xi_{n-s}+1)(Y_{n-s:n} - \hat{\mu}) + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)(Y_{i:n} - \hat{\mu}) \Big] \tag{2.30}$$

where

$$D = r\alpha_{2r} - se^{\xi_{n-s}}(1 - \xi_{n-s}) + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} + e^{\xi_i}\xi_i),$$

$$D' = r\alpha_{1r} + se^{\xi_{n-s}}\xi_{n-s}^2 + \sum_{i=r+1}^{n-s} e^{\xi_i}\xi_i^2 + (n-r-s),$$

$$E = r\beta_{2r} - se^{\xi_{n-s}} - \sum_{i=r+1}^{n-s} e^{\xi_i},$$

$$E' = r\beta_{1r} - se^{\xi_{n-s}}(1 + \xi_{n-s}) + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i),$$

$$F = r\beta_{2r}Y_{r+1:n} - se^{\xi_{n-s}}Y_{n-s:n} - \sum_{i=r+1}^{n-s} e^{\xi_i}Y_{i:n},$$

and

$$F' = r\beta_{1r}Y_{r+1:n} - se^{\xi_{n-s}}(1 + \xi_{n-s})Y_{n-s:n} + \sum_{i=r+1}^{n-s}(1 - e^{\xi_i} - e^{\xi_i}\xi_i)Y_{i:n}.$$

Since $\mu = \ln\theta$ and $\delta = 1/\sigma$, we can obtain the AMLEs of the scale parameter θ and the shape parameter δ of the Weibull distribution with pdf (1.1) as follows;

$$\hat{\theta} = e^{\hat{\mu}} \text{ and } \hat{\delta} = 1/\hat{\sigma}.$$

We simulate MSEs for the proposed estimators of δ and θ by Monte Carlo simulation method. The simulation procedure is repeated 10,000 times for the sample size $n = 15, 20, 30,$ and 50, various choice of censoring.

The MSEs of two proposed estimators of the shape parameter the same, regardless of value of the scale parameter, because the scale parameter for the Weibull distribution is the function of the location parameter for the extreme value distribution. The MSEs of $\hat{\delta}_2$ are smaller than those of $\hat{\delta}_1$, but the MSEs of $\hat{\sigma}_1$ are smaller than those of $\hat{\sigma}_2$. The MSEs of $\hat{\delta}$ and $\hat{\theta}$ based on left censored samples are smaller than those of the estimators based on right censored sample.

Table 1. The relative MSEs of $\hat{\delta}_1$ and $\hat{\delta}_2$ when Θ is known

n		15		20	
r	s	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_1$	$\hat{\delta}_2$
0	0	0.088188	0.053826	0.054065	0.036727
0	1	0.099702	0.062486	0.059802	0.041733
0	2	0.119040	0.073537	0.066740	0.046613
0	3	0.142528	0.087976	0.074769	0.052062
1	0	0.098218	0.058933	0.057949	0.038845
2	0	0.109371	0.064040	0.062061	0.041324
3	0	0.127490	0.072071	0.067308	0.044196
1	1	0.112697	0.069067	0.064418	0.044321
1	2	0.137398	0.081949	0.072905	0.050174
1	3	0.167825	0.099416	0.082647	0.056539
2	1	0.128615	0.076704	0.069651	0.047498
2	2	0.160454	0.093005	0.079701	0.054277
2	3	0.199252	0.114191	0.091098	0.061397
3	1	0.154315	0.088705	0.076247	0.051188
3	2	0.197553	0.110093	0.087879	0.058652
3	3	0.255898	0.138938	0.102275	0.067290
n		30		50	
r	s	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\delta}_1$	$\hat{\delta}_2$
0	0	0.030512	0.022992	0.015205	0.012655
0	1	0.032348	0.024947	0.015922	0.013530
0	2	0.035084	0.027169	0.016675	0.014247
0	3	0.038199	0.029635	0.017595	0.015067
1	0	0.031796	0.023834	0.015523	0.012906
2	0	0.033012	0.024646	0.015891	0.013194
3	0	0.034498	0.025619	0.016185	0.013414
1	1	0.033801	0.025924	0.016250	0.013795
1	2	0.036712	0.028249	0.017028	0.014531
1	3	0.040156	0.030968	0.017979	0.015374
2	1	0.035181	0.026874	0.016652	0.014120
2	2	0.038388	0.029394	0.017473	0.014892
2	3	0.042156	0.032357	0.018452	0.015761
3	1	0.036844	0.027984	0.016960	0.014353
3	2	0.040315	0.030690	0.017818	0.015159
3	3	0.044541	0.034023	0.018842	0.016064

Table 2. The relative MSEs of $\hat{\delta}$ and $\hat{\theta}$

n		15		20	
r	s	$\hat{\delta}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\theta}$
0	0	0.095323	0.107888	0.058610	0.075815
0	1	0.108734	0.113679	0.063849	0.078354
0	2	0.123168	0.123836	0.069482	0.082177
0	3	0.144404	0.139186	0.076952	0.086469
1	0	0.105800	0.106546	0.063007	0.075450
2	0	0.119170	0.105672	0.067792	0.075244
3	0	0.135588	0.104683	0.074198	0.074603
1	1	0.123115	0.112351	0.069034	0.078001
1	2	0.142927	0.122469	0.075452	0.081784
1	3	0.172468	0.137966	0.084201	0.086036
2	1	0.141899	0.111351	0.075169	0.077751
2	2	0.168425	0.121288	0.082511	0.081475
2	3	0.212202	0.136719	0.093062	0.085682
3	1	0.166030	0.110225	0.083322	0.077120
3	2	0.203871	0.119800	0.091632	0.080721
3	3	0.272654	0.134989	0.104262	0.084922
n		30		50	
r	s	$\hat{\delta}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\theta}$
0	0	0.032859	0.044874	0.016590	0.024670
0	1	0.034491	0.045538	0.017198	0.024852
0	2	0.036775	0.046972	0.017753	0.025065
0	3	0.039328	0.048295	0.018338	0.025451
1	0	0.034017	0.044714	0.016935	0.024663
2	0	0.035504	0.044728	0.017294	0.024659
3	0	0.037303	0.044719	0.017686	0.024639
1	1	0.035705	0.045389	0.017571	0.024841
1	2	0.038113	0.046819	0.018135	0.025054
1	3	0.040820	0.048137	0.018729	0.025437
2	1	0.037459	0.045387	0.017952	0.024837
2	2	0.040195	0.046791	0.018569	0.025046
2	3	0.043215	0.048092	0.019191	0.025427
3	1	0.039545	0.045370	0.018401	0.024814
3	2	0.042715	0.046730	0.019064	0.025022
3	3	0.046045	0.048013	0.019743	0.025396

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