An $M/G/1$ queue under the $P^{M}_{\lambda,\tau}$ service policy[1)]

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Abstract

We analyze an $M/G/1$ queueing system under $P^{M}_{\lambda,\tau}$ service policy. By using the level crossing theory and solving the corresponding integral equations, we obtain the stationary distribution of the workload in the system explicitly.

Keywords : $M/G/1$ queue, $P^{M}_{\lambda,\tau}$ policy, stationary distribution

1. Introduction

The $P^{M}_{\lambda,\tau}$ policy was introduced by Yeh(1985) as a generalized releasing policy of the P^{M}_{λ} policy of Faddy(1974) for a dam with input formed by a Wiener process. Abdel-Hameed(2000) considered the optimal control of a dam using $P^{M}_{\lambda,\tau}$ policy when the input process is a compound Poisson process with positive drift. Bae et al.(2003) determined the long-run average cost per unit time under the $P^{M}_{\lambda,\tau}$ policy in a finite dam with a compound Poisson input. Under the P^{M}_{λ} policy, the stationary distribution of the workload in the $M/G/1$ queueing system was derived in Bae et al.(2002).

In this paper, we introduce the $P^{M}_{\lambda,\tau}$ policy for an $M/G/1$ queueing system; a server is initially idle and starts to serve, if a customer arrives, with service speed 1. The customers arrive according to a Poisson process of rate $\nu(>0)$ and each customer brings a job consisting of an amount of work to be processed that is independently and identically distributed with a distribution function G and a mean $m(>0)$. If the workload exceeds threshold $\lambda(>0)$, the server changes his service speed to $M(>1)$ instantaneously and continues to follow that service speed until the workload level reaches $\tau(0<\tau<\lambda)$. When the workload reaches level τ, the service speed is changed again to 1 instantaneously. The service speed 1 is kept until the level up-crosses λ again. For the stability of the

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system, we assume that $\rho = \nu m < 1$.

In this paper, using a similar method as in Bae et al.(2002), we derive the distribution of the workload at the exit time from $(0,\lambda]$. Together with the level crossing theory, it enables us to determine the explicit stationary distribution of the workload.

2. The excess amount over λ at the exit time from $[0,\lambda]$

Let $X(t)$ denote the workload of the system at time t under the $P_{\lambda,\tau}^{M}$ service policy. If we define $T_0^{\lambda} = \inf\{t > 0 \,|\, X(t) > \lambda\}$ and $T_0^{\tau} = \inf\{t > T_0^{\lambda} \,|\, X(t) = \tau\}$, and for $n \ge 1$, $T_n^{\lambda} = \inf\{t > T_{n-1}^{\tau} \,|\, X(t) > \lambda\}$ and $T_n^{\tau} = \inf\{t > T_n^{\lambda} \,|\, X(t) = \tau\}$, then $\{X(t),\, t \ge 0\}$ is a delayed regenerative process having $T_0^{\tau}, T_1^{\tau}, T_2^{\tau}, \cdots$ as regeneration points.

Since $\{X(t),\, t \ge 0\}$ is non-Markovian, we decompose it into two Markov processes. Let $\{X_1(t),\, t \ge 0\}$ be a process obtained from $\{X(t),\, t \ge 0\}$ by deleting the time periods from T_n^{λ} to T_n^{τ}, for all $n \ge 0$, and by gluing together the remaining periods. Note that in the process $\{X_1(t),\, t \ge 0\}$ the system operates with service speed 1. Let $\{X_2(t),\, t \ge 0\}$ be formed similarly by separating and connecting the periods which start at T_n^{λ} and end at T_n^{τ}, for all $n \ge 0$. Then, clearly the process $\{X_2(t),\, t \ge 0\}$ has the service speed M.

Now, we observe the excess amount over λ at the first passage time through λ of the process $\{X(t),\, t \ge 0\}$. Note that it is the same as the excess amount over λ at the end of the cycle of the process $\{X_1(t),\, t \ge 0\}$. Let us denote the exit time of the process $\{X_1(t),\, t \ge 0\}$, starting at x, from $(0,\lambda]$ by T_x, namely,

$$T_x := \inf\{t \ge 0 \,|\, X_1(t) \not\in (0,\lambda], X_1(0) = x\},\ 0 \le x \le \lambda,$$

and define the distribution of the workload at the exit time T_x by

$$Q(l,x) := \Pr\{X_1(T_x) > \lambda + l\}, \qquad l \ge 0,\ 0 \le x \le \lambda.$$

Let

$$K^*(x,y) := \sum_{n=1}^{\infty} K^n(x,y),\ 0 \le y < x,$$

with

$$K^1(x,y) := \nu(1 - G(x-y))$$

and

$$K^{n+1}(x,y) = \int_y^x K^n(x,z) K^1(z,y)\,dz = \int_y^x K^1(x,z) K^n(z,y)\,dz,\ n \ge 1.$$

Then, we obtain the following lemma:

Lemma 1 *For* $l \ge 0$,

$$Q(l,x)=\begin{cases}0, & x=0,\\ \int_0^{\lambda-x}\tilde{q}(l,y)dy+\tilde{Q}(l,0), & 0<x\le\lambda,\end{cases}$$

where

$$\tilde{q}(l,y):=h(l,y)+\int_0^y h(l,z)K^*(y,z)dz,$$

$$h(l,y):=\nu\{\tilde{Q}(l,0)(1-G(y))-(1-G(y+l))\},$$

and

$$\tilde{Q}(l,0):=\frac{\int_0^\lambda \nu(1-G(y+l))dy+\int_0^\lambda\int_0^y \nu(1-G(z+l))K^*(y,z)dzdy}{1+\int_0^\lambda K^*(y,0)dy}.$$

Remark 1 $Q(0,x)$ is the probability that the process $\{X_1(t),\ t\ge 0\}$, starting from $0<x\le\lambda$, up-crosses level λ without reaching level 0 given by

$$Q(0,x)=\frac{\int_{\lambda-x}^{\lambda}K^*(y,0)dy}{1+\int_0^\lambda K^*(y,0)dy}.$$

In the next lemma, we express the distribution of the excess amount over λ at the first passage time through λ in terms of $Q(l,x)$ obtained in Lemma 1.

Lemma 2 *The excess amount over* λ *for the process* $\{X_1(t),\ t\ge 0\}$*, starting with* $x(0\le x\le\lambda)$*, denoted by* L_x*, has the distribution function given by*

$$P(l,x):=\Pr\{L_x\le l\}=1-Q(l,x)+(Q(0,x)-1)\frac{1-G(\lambda+l)+\int_0^\lambda Q(l,x)dG(x)}{1-G(\lambda)+\int_0^\lambda Q(0,x)dG(x)}. \tag{1}$$

3. The stationary distribution

We denote by C, C_1, and C_2 the cycles of the processes $\{X(t),\ t\ge 0\}$, $\{X_1(t),\ t\ge 0\}$, and $\{X_2(t),\ t\ge 0\}$, respectively. Then, obviously $C=C_1+C_2$.

Because $\{X(t),\ t\ge 0\}$ and $\{X_i(t),\ t\ge 0\}$ for $i=1,2$, are regenerative processes with finite mean cycles, each process has its stationary distribution function. Let F_i be the stationary distribution function of $\{X_i(t),\ t\ge 0\}$ for $i=1,2$, and let F be that of $\{X(t),\ t\ge 0\}$. Then it follows that

$$F(x)=\beta F_1(x)+(1-\beta)F_2(x), \tag{2}$$

where $\beta=E[C_1]/E[C]$. Note that F_2 is continuous and supported on $[\tau,\infty)$, whereas F_1 is supported on $[0,\lambda]$, has a jump at zero, and is continuous otherwise. We denote the jump size of F_1 at zero by α and write

$$F_1(x) = \alpha + (1-\alpha)F_1^{ac}(x),$$

where F_1^{ac} is the absolutely continuous part of F_1. Using (2), the distribution F can be written as

$$F(x) = \alpha\beta + (1-\alpha)\beta F_1^{ac}(x) + (1-\beta)F_2(x).$$

For $i = 1,2$, let $D_i(x)$ and $U_i(x)$ be the numbers of down- and up-crossings of level x by the process $\{X_i(t), t \geq 0\}$ during the cycle C_i, respectively, and N_i the number of arrivals during C_i. By convention the arrival that causes $\{X(t),\ t \geq 0\}$ to up-cross level λ for the first time during the cycle C is counted only in N_1.

By using the level crossing theory in Cohen(1977), we have that for the number of down-crossings, for $i = 1,2$,

$$E[D_i(x)] = E[C_i]\frac{d}{dx}F_i(x).$$

We also have that, for $i = 1,2$,

$$E[U_i(x)] = E[N_i]E[1_{\{X_i \leq x\}} - 1_{\{X_i + S \leq x\}}],$$

where X_i is the generic random variable with distributions F_i, for $i = 1,2$, and S denotes the amount of work that each arriving customer carries to the system.

Because the process $\{X(t),\ t \geq 0\}$ is the regenerative process having the same level τ at all regeneration points, the number of up-crossings of level x equals the number of down-crossings of that level during the cycle. Therefore, it follows that

$$D_1(x) = \begin{cases} U_1(x), & 0 < x < \tau, \\ U_1(x) - 1, & \tau \leq x < \lambda, \end{cases}$$

and

$$D_2(x) = \begin{cases} U_2(x) + 1, & \tau < x < \lambda, \\ U_2(x) + U_1(x), & x \geq \lambda, \end{cases} \tag{3}$$

where $U_1(x)$ in (3) means the number of arrivals during the cycle C_1 that cause the process $\{X_1(t),\ t \geq 0\}$ to up-cross both level λ and level $x(\geq \lambda)$ simultaneously.

Let f_1^{ac} and f_2 are densities corresponding to F_1^{ac} and F_2, respectively. Then we have the following theorem:

Theorem 1 *The stationary densities f_1^{ac} and f_2 are given, respectively, by*

$$f_1^{ac}(x) = \begin{cases} \dfrac{\alpha}{1-\alpha}K^*(x,0), & 0 < x < \tau, \\ \dfrac{\alpha}{1-\alpha}\left\{K^*(x,0) - \dfrac{\nu Q(\lambda)}{1-Q(0,\tau)}\left(1 + \displaystyle\int_\tau^x K^*(x,y)dy\right)\right\}, & \tau \leq x < \lambda, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f_2(x)=\begin{cases}\dfrac{\alpha\beta\nu Q(\lambda)}{(1-\beta)M(1-Q(0,\tau))}\left\{1+\displaystyle\int_\tau^x K_M{}^*(x,y)dy\right\}, & \tau<x<\lambda,\\ \dfrac{\alpha\beta\nu Q(\lambda)}{(1-\beta)M(1-Q(0,\tau))}\left\{1-P(x-\lambda,\tau)+\displaystyle\int_\tau^\lambda K_M{}^*(x,y)dy\right. & \\ \quad\left.+\displaystyle\int_\lambda^x(1-P(y-\lambda,\tau))K_M{}^*(x,y)dy\right\}, & x\geq\lambda,\\ 0, & \text{otherwise},\end{cases}$$

where

$$Q(\lambda):=1-G(\lambda)+\int_0^\lambda Q(0,x)dG(x)$$

and

$$K_M{}^*(x,y):=\sum_{n=1}^\infty\frac{K^n(x,y)}{M^n},\qquad 0\leq y<x,$$

and finally α *and* β *are determined by two normalizing conditions*

$$\alpha+(1-\alpha)\int_0^\lambda f_1^{ac}(x)dx=1$$

and

$$\int_\tau^\infty f_2(x)dx=1.$$

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