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Noninformative Priors for Fieller-Creasy Problem using Unbalanced Data

Dal Ho Kim[1], Woo Dong Lee[2], Sang Gil Kang[3]

Abstract

The Fieller-Creasy problem involves statistical inference about the ratio of two independent normal means. It is difficult problem from either a frequentist or a likelihood perspective. As an alternatives, a Bayesian analysis with noninformative priors may provide a solution to this problem.

In this paper, we extend the results of Yin and Ghosh (2001) to unbalanced sample case. We find various noninformative priors such as first and second order matching priors, reference and Jeffreys' priors.

The posterior propriety under the proposed noninformative priors will be given. Using real data, we provide illustrative examples. Through simulation study, we compute the frequentist coverage probabilities for probability matching and reference priors. Some simulation results will be given.

Key Words : Matching Prior; Reference Prior; Fieller-Creasy Problem; Ratio of Normal Means; Unbalanced Data; Likelihood Inference.

1. Introduction

The Fieller-Creasy problem involves statistical inference about the ratio of two independent normal means. It is difficult problem from either a frequentist or a likelihood perspective. As an alternatives, a Bayesian analysis with noninformative priors may provide a solution to this problem.

1)Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea.

2)Department of Asset Managemental Science, Daegu Haany University, Kyungsan, 712-240, Korea.

3)Department of Applied Statistics,Sangji University, Wonju, 220-702, Korea.

Bayesian analysis for the original Fieller-Creasy problem based on noninformative priors began with Kappenman et al. (1970), and was addressed subsequently in Bernardo (1977), Sendra (1982), Mendoza (1996), Stephens and Smith (1992), Liseo (1993), Phillipe and Robert (1994), Reid (1995) and Berger et al (1999). All these papers considered either Jeffreys' prior or reference priors. A Bayesian analysis based on proper priors is given in Carlin and Louis (2000).

Recently, Yin and Ghosh (2001) developed noninformative priors for this problem and studied the Bayesian and likelihood based inference. But their study were too restrictive in the sense that they developed the Bayesian inference under the condition that the sample size of the two population is same. That their results can apply only a balance sample case.
But in the real fields, data with unbalanced cases are common. So, we feel strong necessities to extend their study to unbalanced data.

The present paper focuses on developing noninformative priors for Fieller-Creasy problem. We consider Bayesian priors such that the resulting credible intervals for the ratio of the two normal means have coverage probabilities equivalent to their frequentist counterparts. Although this matching can be justified only asymptotically, our simulation results indicate that this is indeed achieved for small or moderate sample sizes as well.

This matching idea goes back to Welch and Peers (1963). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Mukerjee and Dey (1993), DiCiccio and Stern (1994), Datta and Ghosh (1995a,b, 1996), Mukerjee and Ghosh (1997).

On the other hand, Ghosh and Mukerjee (1992), and Berger and Bernardo (1989,1992) extended Bernardo's (1979) reference prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion described earlier.

In this paper, we extend the results of Yin and Ghosh (2001) to unbalanced sample case. We find various noninformative priors such as first and second order matching priors, reference and Jeffreys' priors.

The posterior propriety under noninformative priors will be given. Using real data, we provide illustrative examples. Through simulation study, we compute the frequentist coverage probabilities for probability matching and reference priors. Some simulation results will be given.

2. Noninformative priors

Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be random samples from $N(\mu, \sigma^2)$ and $N(\theta\mu, \sigma^2)$, respectively. And X_i and Y_j are independently distributed. Here our parameter of interest is θ. Then the likelihood function for (θ, μ, σ^2) is given by

$$\begin{aligned} L(\theta, \mu, \sigma) &= (\sigma\sqrt{2\pi})^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \\ &\quad \times (\sigma\sqrt{2\pi})^{-m} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{m}(y_j-\theta\mu)^2\right\} \\ &\propto \sigma^{-N} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu)^2+\sum_{j=1}^{m}(y_j-\theta\mu)^2\right]\right\}, \end{aligned}$$

where $N = n + m$.

Consider the following transformation for (θ, μ, σ^2).

$$\theta_1 = \theta,\ \theta_2 = \mu(n+m\theta^2)^{1/2} \text{ and } \theta_3 = \sigma^2.$$

Then the likelihood function for $(\theta_1, \theta_2, \theta_3)$ can be re-expressed as

$$\begin{aligned} &L(\theta_1, \theta_2, \theta_3) \propto \theta_3^{-N/2} \\ &\times \exp\left\{-\frac{1}{2\theta_3}\left[\sum_{i=1}^{n}(x_i-\theta_2(n+m\theta_1^2)^{-1/2})^2+\sum_{j=1}^{m}(y_j-\theta_1\theta_2(n+m\theta_1^2)^{-1/2})^2\right]\right\}. \end{aligned}$$

From the above likelihood function, the Fisher information matrix is given by

$$I = \begin{pmatrix} \theta_3^{-1}\theta_2^2 nm(n+m\theta_1^2)^{-2} & 0 & 0 \\ 0 & \theta_3^{-1} & 0 \\ 0 & 0 & \frac{N}{2\theta_3^2} \end{pmatrix}.$$

Following Tibshirani (1989), the class of first order matching prior is given by

$$\pi_M^{(1)}(\theta_1, \theta_2, \theta_3) \propto |\theta_2| \theta_3^{-1/2}(n+m\theta_1^2)^{-1} g(\theta_2, \theta_3),$$

where $g(\cdot, \cdot)$ is an arbitrary positive and differentiable function in its arguments.

Since the class of first order matching prior is quite large, one needs to narrow down this prior. Specially, Murkerjee and Ghosh (1997) developed a second order matching prior. Among the first order matching prior, the second order matching prior satisfies the following differential equation.

$$\frac{1}{6} g(\theta_2, \theta_3) \frac{\partial}{\partial \theta_1}\{I_{11}^{-3/2} L_{1,1,1}\} + \sum_{v=2}^{3}\sum_{s=2}^{3}\{I_{11}^{-1/2} L_{11s} I^{sv} g(\theta_2, \theta_3)\} = 0,$$

where I_{ij} is the i-th and j-th element of Fisher information matrix, I^{sv} is the s-th and v-th element of inverse of Fisher information matrix,

$$L_{1,1,1} = E\left[\left(\frac{\partial \log L(\theta_1, \theta_2, \theta_3)}{\partial \theta_1}\right)^3\right]$$

and

$$L_{ijk} = E\left[\frac{\partial^3 \log L(\theta_1, \theta_2, \theta_3)}{\partial \theta_i \partial \theta_j \partial \theta_k}\right].$$

From the above likelihood function and after long algebraic calculation, one can get

$$I^{22} = \theta_3, I^{23} = I^{32} = 0, I^{33} = \frac{2\theta_3^2}{n+m},$$

$$L_{1,1,1} = 0, L_{112} = -nm\,\theta_2\theta_3^{-1}(n+m\theta_1^2)^{-2} \text{ and}$$

$$L_{113} = nm\,\theta_2^2\theta_3^{-2}(n+m\theta_1^2)^{-2}.$$

Then the differential equation reduces to

$$-\theta_3^{1/2}\frac{\partial}{\partial \theta_2} g(\theta_2, \theta_3) + \frac{2}{n+m}\frac{\partial}{\partial \theta_3}\theta_2\theta_3^{1/2} g(\theta_2, \theta_3) = 0.$$

A solution of the above equation is

$$g(\theta_2, \theta_3) = \theta_3^{-1/2} h(\frac{\theta_2^2}{n+m} + \theta_3),$$

where $h(\cdot)$ is an arbitrary positive differentiable function in its arguments. So, if one takes $h(\frac{\theta_2^2}{n+m} + \theta_3) = 1$, then the second order probability matching prior is given by

$$\pi_M^{(2)}(\theta_1, \theta_2, \theta_3) = |\theta_2| \theta_3^{-1}(n + m\theta_1^2)^{-1}.$$

Remark 1. The second order matching prior given in the above is not an alternative matching prior introduced by Mukerjee and Reid (1999). They suggested the conditions which can verify whether a second order matching prior satisfied an alternative coverage matching prior or not. But among their conditions, a second matching prior satisfies the following equation to be an alternative
coverage matching prior. The equation is

$$I_{11}^{-3/2} L_{111} = 0.$$

But in our case,

$$L_{111} = \frac{6m^2 n\theta_1\theta_2^2}{(n + m\theta_1^2)^3\theta_3}.$$

So,

$$I_{11}^{-3/2} L_{111} = 6n^{-1}m^{1/2}\theta_1\theta_2^{-1}\theta_3^{1/2}.$$

This leads to

$$\frac{\partial}{\partial\theta_1}(I_{11}^{-3/2} L_{111}) = 6n^{-1}m^{1/2}\theta_2^{-1}\theta_3^{1/2} \neq 0.$$

Remark 2. Datta, Ghosh and Mukerjee (2000) showed that if $I_{11}^{-3/2} L_{111}$ does not depend on θ_1, then the second order matching prior is highest posterior distribution (HPD) matching prior within the first order matching priors. So, we can conclude that the second order matching prior is not a HPD matching prior.

Following Datta and Ghosh (1995), the reference prior introduced by Berger and Bernardo (1989) can be obtained easily from the information matrix, if parameters orthogonality is satisfied.
From the information matrix, the reference priors by the order of inferential importance are give as follows:

the order of importance	reference priors
$(\{\theta_1\},\{\theta_2\},\{\theta_3\})$	$\pi_R^1(\theta_1,\theta_2,\theta_3)\propto(n+m\theta_1^2)^{-1}\theta_3^{-1}$
$(\{\theta_1,\theta_2\},\{\theta_3\})$	$\pi_R^2(\theta_1,\theta_2,\theta_3)\propto(n+m\theta_1^2)^{-1}\theta_3^{-1}\mid\theta_2\mid$
$(\{\theta_1,\theta_2,\theta_3\})$	$\pi_R^3(\theta_1,\theta_2,\theta_3)\propto(n+m\theta_1^2)^{-1}\theta_3^{-2}\mid\theta_2\mid$
$(\{\theta_1,\{\theta_2,\theta_3\}),(\{\theta_1,\theta_3\},\{\theta_2\}),(\{\theta_2,\theta_3\},\{\theta_1\})$	$\pi_R^4(\theta_1,\theta_2,\theta_3)\propto(n+m\theta_1^2)^{-1}\theta_3^{-3/2}$

Note that, the prior π_R^1 is called the one-at-a-time reference prior. The two group reference prior π_R^2 is actually the second order matching prior. And π_R^3 is Jeffreys' prior.

3. Propriety of Posteriors

In this section, we will show the propriety of posterior distributions induced by various noninformative priors given in the previous section. The noninformative priors proposed in the previous section can be represented as a general form as follows.

$$\pi_G(\theta_1,\theta_2,\theta_3)\propto(n+m\theta_1^2)^{-1}\mid\theta_2\mid^a\theta_3^{-b},$$

where $a=0,1$ and $b=1/2,1,3/2,2$.

Using the above prior, the joint posterior of θ_1, θ_2 and θ_3 is given by

$$\pi_G(\theta_1,\theta_2,\theta_3\mid \underline{x},\underline{y})\propto(n+m\theta_1^2)^{-1}\mid\theta_2\mid^a\theta_3^{-(N/2+b)}$$
$$\times\exp\left\{-\frac{1}{2\theta_3}\left[s_x+s_y+n(\overline{x}-\theta_2(n+m\theta_1^2)^{-1/2})^2+m(\overline{y}-\theta_1\theta_2(n+m\theta_1^2)^{-1/2})^2\right]\right\},$$

where $s_x=\sum_{i=1}^n(x_i-\overline{x})^2$ and $s_y=\sum_{i=1}^n(y_i-\overline{y})^2$.

Let $\theta=\theta_1,\mu=\theta_2(n+m\theta_1^2)^{-1/2}$ and $\tau=\theta_3^{-1}$, then the above joint posterior changes to

$$\pi_G(\theta,\mu,\tau\mid \underline{x},\underline{y})\propto\mid\mu\mid^a(n+m\theta^2)^{\frac{a-1}{2}}\tau^{\frac{N}{2}+b-2}\exp\left\{-\frac{\tau}{2}\left[s_x+s_y+n(\overline{x}-\mu)^2+m(\overline{y}-\theta\mu)^2\right]\right\}.$$

Now, we will consider the propriety of the joint posterior distribution.

Theorem 1. If $\frac{N}{2}+b-\frac{3}{2}>0$, then the joint posterior distribution of θ, μ and τ is proper.

Proof. For the convenance, we consider the proof with respect to the values of a

i) First, when $a=0$, the posterior is given by

$$\pi_G(\theta,\mu,\tau \mid \underline{x},\underline{y}) \propto (n+m\theta^2)^{-\frac{1}{2}}\tau^{\frac{N}{2}+b-2}\exp\left\{-\frac{\tau}{2}[s_x+s_y+n(\overline{x}-\mu)^2+m(\overline{y}-\theta\mu)^2]\right\}.$$

Integration with respect to μ is

$$\pi_G(\theta,\mu \mid \underline{x},\underline{y}) \propto (n+m\theta^2)^{-1}\tau^{\frac{N}{2}+b-\frac{5}{2}}\exp\left\{-\frac{\tau}{2}\left(s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right)\right\}.$$

And the integration with respect to τ is, if $\frac{N}{2}+b-\frac{3}{2}>0$,

$$\begin{aligned}\pi_G(\theta \mid \underline{x},\underline{y}) &\propto (n+m\theta^{2)^{-1}}\left[s_x+s_y+\frac{nm(\overline{y}-\theta\overline{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-\frac{3}{2})} \\ &\propto (n+m\theta^2)^{-1}\left[1+\frac{nm(\overline{y}-\theta\overline{x})^2}{(n+m\theta^2)(s_x+s_y)}\right]^{-(\frac{N}{2}+b-\frac{3}{2})} \\ &\le (n+m\theta^2)^{-1},\end{aligned}$$

since

$$\left[1+\frac{nm(\overline{y}-\theta\overline{x})^2}{(n+m\theta^2)(s_x+s_y)}\right]^{-(\frac{N}{2}+b-\frac{3}{2})} \le 1.$$

Now the integration with respect to θ is

$$\int_{-\infty}^{\infty}\frac{1}{n+m\theta^2}\,d\theta=\frac{\sqrt{nm}}{\pi}.$$

Therefore, posterior is proper when $a=0$.

ii) When $a=1$, the joint posterior is given by

$$\begin{aligned}\pi_G(\theta,\mu,\tau\mid \underline{x},\underline{y}) &\propto \mid\mu\mid \tau^{\frac{N}{2}+b-2}\exp\left\{-\frac{\tau}{2}\left[s_x+s_y+n(\bar{x}-\mu)^2+m(\bar{y}-\theta\mu)^2\right]\right\}\\ &\propto \mid\mu\mid \exp\left\{-\frac{\tau(n+m\theta^2)}{2}\left(\mu-\frac{n\bar{x}+\theta m\bar{y}}{n+m\theta^2}\right)^2\right\}\\ &\quad\times\tau^{\frac{N}{2}+b-2}\exp\left\{-\frac{\tau}{2}\left(s_x+s_y+\frac{nm(\bar{y}-\theta\bar{x})^2}{n+m\theta^2}\right)\right\}.\end{aligned}$$

Note that

$$\int_{-\infty}^{\infty}\mid x\mid \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}dx=\sigma^2\left(2\exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}+\sqrt{2\pi}\sqrt{\frac{\mu^2}{\sigma^2}}\,Erf\left[\sqrt{\frac{\mu^2}{2\sigma^2}}\right]\right),$$

where $Erf(a)=\int_0^a\frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}dx$, with $a>0$.

Using the above equation, the integration with respect to μ is

$$\begin{aligned}&\int_{-\infty}^{\infty}\mid\mu\mid\exp\left\{-\frac{\tau(n+m\theta^2)}{2}\left(\mu-\frac{n\bar{x}+\theta m\bar{y}}{n+m\theta^2}\right)^2\right\}d\mu\\ &=[\tau(n+m\theta^2)]^{-1}\left\{2\exp\left\{-\frac{\tau(n\bar{x}+\theta m\bar{y})^2}{2(n+m\theta^2)}\right\}+\sqrt{2\pi}\sqrt{\frac{\tau(n\bar{x}+\theta m\bar{y})^2}{2(n+m\theta^{2)}}}\,Erf\left(\sqrt{\frac{\tau(n\bar{x}+\theta m\bar{y})^2}{2(n+m\theta^{2)}}}\right)\right\}.\end{aligned}$$

Since $\tau>0$, $\exp\left\{-\frac{\tau(n\bar{x}+\theta m\bar{y})^2}{2(n+m\theta^2)}\right\}\le 1$ and $Erf(\cdot)\le 1$, the joint posterior distribution of θ and τ is bounded by

$$\begin{aligned}\pi_G(\theta,\tau\mid\underline{x},\underline{y}) &\le \frac{\tau^{\frac{N}{2}+b-3}}{n+m\theta^2}\left\{2+\sqrt{2\pi}\sqrt{\frac{\tau(n\bar{x}+m\theta\bar{y})^2}{(n+m\theta^2)}}\right\}\exp\left\{-\frac{1}{2}\tau\left(s_x+s_y+\frac{nm(\bar{y}-\theta\bar{x})^2}{n+m\theta^2}\right)\right\}\\ &=2\frac{\tau^{\frac{N}{2}+b-3}}{n+m\theta^2}\exp\left\{-\frac{1}{2}\tau\left(s_x+s_y+\frac{nm(\bar{y}-\theta\bar{x})^2}{n+m\theta^2}\right)\right\}\\ &\quad+\sqrt{2\pi}\frac{\tau^{\frac{N}{2}+b-\frac{5}{2}}}{n+m\theta^2}\sqrt{\frac{(n\bar{x}+m\theta\bar{y})^2}{(n+m\theta^2)}}\exp\left\{-\frac{1}{2}\tau\left(s_x+s_y+\frac{nm(\bar{y}-\theta\bar{x})^2}{n+m\theta^2}\right)\right\}.\end{aligned}$$

Integration with respect to τ in the right side of the last equality is proportional to

$$\begin{aligned}&(n+m\theta^2)^{-1}\left[s_x+s_y+\frac{nm(\bar{y}-\theta\bar{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-2)}\\ &+(n+m\theta^2)^{-1}\left[s_x+s_y+\frac{nm(\bar{y}-\theta\bar{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-\frac{3}{2})}\sqrt{\frac{(n\bar{x}+m\theta\bar{y})^2}{n+m\theta^2}}.\end{aligned}$$

Now, the first term of the above quantity is proportional to

$$(n+m\theta^2)^{-1}\left[s_x+s_y+\frac{nm(\bar{y}-\theta\bar{x})^2}{n+m\theta^2}\right]^{-(\frac{N}{2}+b-2)}$$
$$\propto(n+m\theta^2)^{-1}\left[1+\frac{nm(\bar{y}-\theta\bar{x})^2}{(n+m\theta^2)(s_x+s_y)}\right]^{-(\frac{N}{2}+b-2)}$$
$$\leq(n+m\theta^2)^{-1},$$

Integration with restpect to θ of the right side of the last inequality is finite as is shown. And the second term is proportional to

$$\left[1+\frac{nm(\bar{y}-\theta\bar{x})^2}{(n+m\theta^2)(s_x+s_y)}\right]^{-(\frac{N}{2}+b-\frac{3}{2})}\frac{|n\bar{x}+m\theta\bar{y}|}{(n+m\theta^2)^{\frac{3}{2}}}$$
$$\leq\frac{|n\bar{x}+m\theta\bar{y}|}{(n+m\theta^2)^{\frac{3}{2}}}$$
$$\leq\frac{|n\bar{x}|}{(n+m\theta^2)^{\frac{3}{2}}}+\frac{|m\theta\bar{y}|}{(n+m\theta^2)^{\frac{3}{2}}}.$$

Since

$$\int_{-\infty}^{\infty}\frac{1}{(n+m\theta^2)^{\frac{3}{2}}}d\theta=\frac{2}{n\sqrt{m}}\quad\text{and}\quad\int_{-\infty}^{\infty}\frac{|\theta|}{(n+m\theta^2)^{\frac{3}{2}}}d\theta=\frac{2}{m\sqrt{n}},$$

this completes the proof.

Theorem 2. Under the pror π_G, the marginal posterior density function of θ is given by, for $-\infty<\theta<\infty$,

$$\pi_G(\theta\mid\underline{x},\underline{y})=\frac{\int_{-\infty}^{\infty}\frac{|\mu|^a(n+m\theta^2)^{\frac{a-1}{2}}}{[s_x+s_yn(\bar{x}-\mu)^2+m(\bar{y}-\theta\mu)^2]^{\frac{N}{2}+b-1}}d\mu}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\mu|^a(n+m\theta^2)^{\frac{a-1}{2}}}{[s_x+s_yn(\bar{x}-\mu)^2+m(\bar{y}-\theta\mu)^2]^{\frac{N}{2}+b-1}}d\mu d\theta},$$

if $\frac{N}{2}+b-\frac{3}{2}>0$.

Proof. It is straightforwards.

4. Simulation results

In this section, we perform some simulations to show the frequentist coverage probabilities with respect to the priors given in the previous section. We estimate the frequentist coverage probability by investigating the credible interval of the marginal posterior density of θ under the proposed priors π_G for several values of θ, n and m. We want to show that the frequentist coverage of a $(1-\alpha)th$ posterior quantile should be close to $1-\alpha$.

The computation of the coverage probability is performed by the following method. First, we fix the values for μ, σ and θ. For prespecified value α, here α is 0.05 (0.95), let $\theta^\pi(\alpha \mid \underline{X}, \underline{Y})$ be the posterior α-quantile of θ given $\underline{X}$ and $\underline{Y}$. That is $F^\pi(\theta^\pi(\alpha \mid \underline{X}, \underline{Y}) \mid \underline{X}, \underline{Y}) = \alpha$, where $F^\pi(\cdot \mid \underline{X}, \underline{Y})$ is the marginal posterior distribution function of θ under the prior π. Then the frequentist coverage probability of this one sided credible interval of θ is

$$P_{(\theta,\mu,\sigma)}(\alpha;\theta) \equiv P_{(\theta,\mu,\sigma)}(\theta \le \theta^\pi(\alpha \mid \underline{X}, \underline{Y})).$$

The estimated $P_{(\theta,\mu,\sigma)}(\alpha;\theta)$ when $\alpha = 0.05(0.95)$ is shown in Table 1. For fixed n, m and (θ,μ,σ), we take 10,000 independent random samples of $\underline{X}$ and $\underline{Y}$. Note that under the prior π for fixed $\underline{X}$ and $\underline{Y}$,

$$\theta \le \theta^\pi(\alpha \mid \underline{X}, \underline{Y}) \quad \Leftrightarrow \quad F^\pi(\theta^\pi(\alpha \mid \underline{X}, \underline{Y}) \mid \underline{X}, \underline{Y}) \le \alpha.$$

So, under the prior π, $P_{(\theta,\mu,\sigma)}(\alpha;\theta)$ can be estimated by the relative frequency of $F^\pi(\theta^\pi(\alpha \mid \underline{X}, \underline{Y}) \mid \underline{X}, \underline{Y}) \le \alpha$. The results are given in Table 1.

It is clear from the table that the second order matching prior performs better than any other priors in matching the target coverage probabilities. And the reference prior π_R^A is comparable to the second order matching prior.

It appears also from our results that when $|\mu| = 0.1$, the values of the frequentist coverage probabilities are far from target probabilities. The poor performance of all the priors for certain regions of the parameter value is not very surprising. Gleser and Hwang (1987, Theorem 1) show that based on any sample of arbitrary but fixed size n, there is a positive probability that confidence interval is infinite set. In our case, this poor performance happens when $|\mu| \approx 0$.

Table 1. The estimated coverage probabilities

		$\mu=0.1$		$\sigma=0.5$		$\theta=0.1$			
n	m	First-Order $a=1, b=0.5$		Jeffreys $a=1, b=2$		Reference $a=0, b=1.5$		Second-Order $a=1, b=1$	
		0.05	0.95	0.05	0.95	0.05	0.95	0.05	0.95
5	5	0.0033	0.9933	0.0114	0.9814	0.0049	0.9908	0.0048	0.9894
5	10	0.0039	0.9921	0.0050	0.9856	0.0023	0.9937	0.0030	0.9910
10	5	0.0022	0.9933	0.0087	0.9849	0.0040	0.9922	0.0053	0.9897
10	10	0.0038	0.9921	0.0073	0.9863	0.0032	0.9935	0.0047	0.9900
10	15	0.0032	0.9925	0.0046	0.9894	0.0022	0.9944	0.0036	0.9914
15	10	0.0051	0.9913	0.0078	0.9876	0.0044	0.9936	0.0058	0.9904
15	15	0.0035	0.9888	0.0048	0.9846	0.0029	0.9910	0.0037	0.9878
15	20	0.0063	0.9869	0.0062	0.9878	0.0031	0.9932	0.0048	0.9900
20	15	0.0042	0.9910	0.0087	0.9836	0.0048	0.9894	0.0078	0.9855
20	20	0.0075	0.9862	0.0106	0.9825	0.0055	0.9896	0.0083	0.9853

		$\mu=1.0$		$\sigma=0.5$		$\theta=0.1$			
n	m	First-Order $a=1, b=0.5$		Jeffreys $a=1, b=2$		Reference $a=0, b=1.5$		Second-Order $a=1, b=1$	
		0.05	0.95	0.05	0.95	0.05	0.95	0.05	0.95
5	5	0.0385	0.9635	0.0733	0.9327	0.0545	0.9483	0.0483	0.9516
5	10	0.0416	0.9618	0.0641	0.9414	0.0497	0.9535	0.0489	0.9539
10	5	0.0383	0.9596	0.0572	0.9395	0.0471	0.9493	0.0483	0.9530
10	10	0.0442	0.9525	0.0596	0.9403	0.0531	0.9462	0.0501	0.9480
10	15	0.0431	0.9588	0.0565	0.9481	0.0487	0.9535	0.0494	0.9547
15	10	0.0453	0.9584	0.0575	0.9466	0.0519	0.9519	0.0493	0.9547
15	15	0.0441	0.9498	0.0526	0.9425	0.0483	0.9469	0.0491	0.9476
15	20	0.0483	0.9525	0.0547	0.9439	0.0504	0.9483	0.0496	0.9502
20	15	0.0460	0.9545	0.0563	0.9466	0.0513	0.9514	0.0497	0.9515
20	20	0.0462	0.9486	0.0521	0.9409	0.0492	0.9457	0.0498	0.9507

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