Proceedings of Korean Data And Information Science Society,
October 28-29, 2005, pp. 27-34

Bayesian Inference for Stress-Strength Systems

In Hong Chang[1], Byung Hwee Kim[2]

Abstract

We consider the problem of estimating the system reliability noninformative priors when both stress and strength follow generalized gamma distributions. We first derive Jeffreys' prior, group ordering reference priors, and matching priors. We investigate the propriety of posterior distributions and provide marginal posterior distributions under those noninformative priors. We also examine whether the reference priors satisfy the probability matching criterion.

Keywords : Generalized gamma distribution, Jeffreys' prior, probability matching prior, orthogonal reparametrization, reference prior, system reliability.

1. Introduction

Suppose a system, made up of k identical components, functions if r or more of the k components simultaneously operate. We assume that the strengths of these components $Y_1, \cdots, Y_k$ are independently and identically distributed(i.i.d.) random variables with a common cumulative distribution function(c.d.f), $G(y)$. We further suppose that this system is subject to a stress, say X, which a random variable with c.d.f. $F(x)$. The system operates satisfactorily if r of more the k components have strength lager than the stress X, and accordingly, we define the system reliability, say $R_{r,k}$, as the probability that at least r of $Y_1, \cdots, Y_k$ exceed X, so that

1) Assistant Professor, Department of Computer Science and Statistics, Chosun University, Gwangju, 501-759, Korea.
E-mail: ihchang@chosun.ac.kr

2) Professor, Department of Mathematics, Hanyang University, Seoul, 133-791, Korea

$$R_{r,k} = P(r \leq \sum_{i=1}^{k} I(X < Y_i))$$

$$= \sum_{i=r}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1-G(x)]^i [G(x)]^{k-i} dF(x) \ . \tag{1.1}$$

In addition, the particular cases $r=1$ and $r=k$ correspond, respectively, to parallel and series systems. The problem of making inference about (1.1) has been discussed using the classical frequentist theory approach, in various guises by Bhattacharyya and Johnson(1974) and Reiser and Guttman(1989), among others. A great deal of this work have focused on producing maximum likelihood estimators, uniformly minimum variance unbiased estimators, one-sided confidence intervals for $R_{r,k}$ in various situations. In contrast, there is relatively little on a Bayesian approach to this problem. Some pertinent references are Draper and Guttman(1978), Guttman et al.(1990), Guttman and Papandonators(1997).

The present paper focuses exclusively on Bayesian for $R_{r,k}$ when $F(x)$ and $G(y)$ are c.d.f.'s of generalized gamma distributions $GG(\eta_1, \beta, p)$ and $GG(\eta_2, \beta, p)$ respectively, with corresponding density functions

$$f(x) = \frac{\beta}{\Gamma(p)} \eta_1^{-p\beta} x^{p\beta-1} e^{-\left(\frac{x}{\eta_1}\right)^{\beta}} \quad x > 0$$

and

$$g(y) = \frac{\beta}{\Gamma(p)} \eta_2^{-p\beta} y^{p\beta-1} e^{-\left(\frac{y}{\eta_2}\right)^{\beta}} \quad y > 0, \tag{1.2}$$

with $\eta_1 > 0$, $\eta_2 > 0$, $\beta > 0$, and $p > 0$. In this situation, the system reliability $R_{r,k}$ in (1.1) reduces, after some manipulation, to

$$R_{r,k} = \sum_{i=r}^{k} \binom{k}{i} \int_{0}^{\infty} [1-I(p,u)]^i [I(p,u)]^{k-i} \frac{1}{\Gamma(p)} \theta_1^p u^{p-1} e^{-\theta_1 u} du \ , \tag{1.3}$$

where $\theta_1 = \left(\frac{\eta_2}{\eta_1}\right)^{\beta}$ and $I(p,u) = \int_0^u \frac{1}{\Gamma(p)} v^{p-1} e^{-v} dv$.

In the generalized gamma distribution $GG(\eta, \beta, p), \eta, \beta$ and p are, respectively, called the scale parameter, the shape parameter, and the index parameter. This distribution includes many interesting distributions as special cases : exponential distribution($p=\beta=1$), Raleigh distribution($p=1, \beta=2$), Weibull distribution($p=1$), Maxwell distribution($p=\frac{3}{2}, \beta=2$), half-normal distribution($p=\frac{1}{2}, \beta=2$), and gamma distribution($\beta=1$).

In this paper, we only consider the case when p is known. Since $R_{r,k}$ in (1.3) depend on θ_1, the emphasis is on noninformative priors for θ_1. Tibshirani(1989) reconsidered the case when the real-valued parameter is orthogonal to the nuisance parameter vector in the sense of Cox and Reid(1987). These priors, as usually referred to as matching priors, were further studied in Datta and Ghosh(1995). In the case of $k=1$, Thompson and Basu(1993) derived reference prior for $R_{1,1}$ when the stress and strength are both exponentially distributed. It turns out that in such situations, the reference prior agrees with Jeffreys' prior.

In this paper we drive matching priors as well as reference priors for θ_1 in generalized gamma stress-strength models when p is known and η_1, η_2, β are unknown parameters. Section 2 treats orthogonal reparameterization from Fisher information matrix of (η_1, η_2, β) to $(\theta_1, \theta_2, \theta_3)$ when θ_1 is the parameter of interest, and provide Fisher information matrix of $(\theta_1, \theta_2, \theta_3)$. In Section 3, we derive, using the Fisher information matrix of $(\theta_1, \theta_2, \theta_3)$, Jeffreys' prior, group ordering reference priors, and matching priors when θ_1 is the parameter of interest. The sufficient condition for propriety of posterior distributions of $(\theta_1, \theta_2, \theta_3)$ and marginal posterior densities of θ_1 under these priors are given in Section 4.

2. Fisher Information Matrix

Suppose that $X_1, \cdots, X_m$ are i.i.d. as the generalized gamma distribution, $GG(\eta_1, \beta, p)$ and independently, $Y_1, \cdots, Y_n$ are i.i.d. as $GG(\eta_2, \beta, p)$.

Then the likelihood function (η_1, η_2, β) is

$$L(\eta_1,\eta_2,\beta\,|\,\underline{x}\,,\underline{y}\,) = \beta^{m+n}\eta_1^{-mp\beta}\eta_2^{-np\beta}\left(\prod_{i=1}^{m}x_i\prod_{j=1}^{n}y_j\right)^{p\beta-1}\cdot e^{-\sum_{i=1}^{m}\left(\frac{x_i}{\eta_1}\right)^{\beta}-\sum_{j=1}^{n}\left(\frac{y_j}{\eta_2}\right)^{\beta}} . \qquad (2.1)$$

The log-likelihood function of (η_1,η_2,β) is

$$\begin{aligned} l(\eta_1,\eta_2,\beta\,|\,\underline{x}\,,\underline{y}\,) &= \log L(\eta_1,\eta_2,\beta\,|\,\underline{x}\,,\underline{y}\,) \\ &\propto (m+n)\log\beta - mp\beta\log\eta_1 - np\beta\log\eta_2 \\ &\quad + (p\beta-1)\left(\sum_{i=1}^{m}\log x_i + \sum_{j=1}^{n}\log y_j\right) - \sum_{i=1}^{m}\left(\frac{x_i}{\eta_1}\right)^{\beta} - \sum_{j=1}^{n}\left(\frac{y_j}{\eta_2}\right)^{\beta} . \end{aligned}$$

Lemma 2.1. The Fisher information matrix of (η_1,η_2,β) is

$$I_1(\eta_1,\eta_2,\beta) = \begin{pmatrix} \frac{mp\beta^2}{\eta_1^2} & 0 & -\frac{mr_1}{\eta_1\Gamma(p)} \\ 0 & \frac{np\beta^2}{\eta_2^2} & -\frac{nr_1}{\eta_2\Gamma(p)} \\ -\frac{mr_1}{\eta_1\Gamma(p)} & -\frac{nr_1}{\eta_2\Gamma(p)} & \frac{(m+n)}{\beta^2}\left(1+\frac{r_2}{\Gamma(p)}\right) \end{pmatrix}, \qquad (2.2)$$

where $r_i = \int_0^{\infty}(\log z)^i z^p e^{-z}dz$, $i=1,2,$

Lemma 2.2. The Fisher information matrix for $(\theta_1,\theta_2,\theta_3)$ is

$$I_3(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} i_{11} & 0 & 0 \\ 0 & i_{22} & 0 \\ 0 & 0 & i_{33} \end{pmatrix} , \qquad (2.3)$$

where

$$i_{11} = mn(m+n)pr_*\theta_1^{-2}[mnp(\log\theta_1)^2 + (m+n)^2 r_*]^{-1} ,$$

$$i_{22} = (m+n)pr_*\theta_2^{-2}\theta_3^2[mnp(\log\theta_1)^2 + (m+n)^2 r_*] ,$$

and

$$i_{33} = \frac{1}{m+n}[mnp(\log\theta_1)^2 + (m+n)^2 r_*]\theta_3^{-2} .$$

This implies that θ_1, the parameter of interest, is orthogonal to the nuisance parameter vector (θ_2,θ_3) in the sense of Cox and Reid(1987) .

3. Noninformative Priors

In this Section, we provide, using (2.3), three types of noninformative priors : Jeffreys' prior, reference priors, and matching priors.

The following theorem gives the reference prior distributions for different groups of ordering for $(\theta_1, \theta_2, \theta_3)$ when θ_1 is the parameter of interest.

Theorem 3.1. If θ_1 is the parameter of interest, then the reference prior distributions for different groups of ordering for $(\theta_1, \theta_2, \theta_3)$ are :

Group ordering	Reference prior
$\{(\theta_1, \theta_2, \theta_3)\}$	$\pi_1(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-1}\theta_2^{-1}[mnp(\log\theta_1)^2+(m+n)^2 r_*]^{\frac{1}{2}}$
$\{\theta_1, (\theta_2, \theta_3)\}$	$\pi_2(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-1}\theta_2^{-1}[mnp(\log\theta_1)^2+(m+n)^2 r_*]^{-\frac{1}{2}}$
$\{(\theta_1, \theta_2), \theta_3\}, \{(\theta_1, \theta_3), \theta_2\}$	$\pi_3(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-1}\theta_2^{-1}\theta_3^{-1}$
$\{\theta_1, \theta_2, \theta_3\}, \{\theta_1, \theta_3, \theta_2\}$	$\pi_4(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-1}\theta_2^{-1}\theta_3^{-1}[mnp(\log\theta_1)^2+(m+n)^2 r_*]^{-\frac{1}{2}}$

Also, following Tibishirani(1989), we have matching priors for θ_1, the parameter of interest, as follows:

Theorem 3.2. The matching priors for θ_1 are given by

$$\pi_M(\theta_1,\theta_2,\theta_3) \propto i_{11}^{\frac{1}{2}} g(\theta_2,\theta_3)$$

$$\propto \theta_1^{-1}[mnp(\log\theta_1)^2+(m+n)^2 r_*]^{-\frac{1}{2}} g(\theta_2,\theta_3) \quad , \qquad (3.1)$$

for any positive differentiable function g.

An interesting class of matching priors can be obtained by taking $g(\theta_2,\theta_3) = \theta_2^{-a}\theta_3^{-b}, a \geq 1, b \leq 2,$ for which (3.1) becomes

$$\pi_M(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-1}\theta_2^{-a}\theta_3^{-b}[mnp(\log\theta_1)^2+(m+n)^2 r_*]^{-\frac{1}{2}} \quad . \qquad (3.2)$$

Note that Jeffreys' prior is same as the reference prior π_1 for $(\theta_1, \theta_2, \theta_3)$ and among the reference priors developed in Theorem 3.1, π_2 and π_4 are the matching prior with respectively a=1, b=0 and a=1, b=1 in (3.2).

4. Posterior Distributions

The posterior density of $(\theta_1, \theta_2, \theta_3)$ under a prior π is

$$\pi(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}) \propto L(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y}) \pi(\theta_1, \theta_2, \theta_3), \tag{4.1}$$

where $L(\theta_1, \theta_2, \theta_3 | \underline{x}, \underline{y})$ is the likelihood function $L(\eta_1, \eta_2, \beta | \underline{x}, \underline{y})$ in (2.1) expressed in terms of

$$\theta_1 = \left(\frac{\eta_2}{\eta_1}\right)^{\beta}, \ \theta_2 = \eta_1^{\frac{m}{m+n}} \eta_2^{\frac{n}{m+n}} e^{\frac{r_1}{p\Gamma(p)} \frac{1}{\beta}},$$

$$\theta_3 = \beta [mnp(\beta \log \frac{\eta_2}{\eta_1})^2 + (m+n)^2 r_*]^{-\frac{1}{2}} .$$

We first provide the sufficient condition under which the posteriors are proper under π_1, π_2, π_3, π_4, and π_M in (3.2). Note that for almost all samples from a continuous distribution, observations are distinct.

Theorem 4.1. All the posterior under π_1, π_2, π_3, π_4, and π_M in (3.2) are proper if $m+n \geq 3$ and $a \geq 1, b \leq 2$.

Next, we provide the marginal posterior densities of θ_1 under π_1, π_2, π_3, π_4, and π_M

Theorem 4.2. Under the priors π_1, π_2, π_3, π_4, and π_M for $a \geq 1, b \leq 2$, the marginal posterior densities of $\theta_1 = \left(\frac{\eta_2}{\eta_1}\right)^{\beta}$ are, respectively, given by

$$\pi_1(\theta_1 | \underline{x}, \underline{y}) \propto \theta_1^{mp-1} \int_0^{\infty} u_3^{m+n-1} h(\theta_1, u_3 | \underline{x}, \underline{y}) du_3 ,$$

$$\pi_2(\theta_1 | \underline{x}, \underline{y}) \propto \theta_1^{mp-1} [mnp(\log \theta_1)^2 + (m+n)^2 r_*]^{-1} \times \int_0^{\infty} u_3^{m+n-1} h(\theta_1, u_3 | \underline{x}, \underline{y}) du_3 ,$$

$$\pi_3(\theta_1 | \underline{x}, \underline{y}) \propto \theta_1^{mp-1} \int_0^{\infty} u_3^{m+n-2} h(\theta_1, u_3 | \underline{x}, \underline{y}) du_3,$$

$$\pi_4(\theta_1 | \underline{x}, \underline{y}) \propto \theta_1^{mp-1} [mnp(\log \theta_1)^2 + (m+n)^2 r_*]^{-\frac{1}{2}}$$

$$\times \int_0^\infty u_3^{m+n-1} h(\theta_1, u_3 | \underline{x}, \underline{y}) du_3 \ ,$$

$$\pi_M(\theta_1 | \underline{x}, \underline{y}) \ \propto \ [mnp(\log\theta_1)^2 + (m+n)^2 r_*]^{\frac{b}{2}-1} \int_0^\infty \theta_1^{mp - \frac{m}{m+n}\frac{1-a}{u_3} - 1}$$

$$\times e^{\frac{r_1}{p\Gamma(p)}\frac{1-a}{u_3}} \Gamma[(m+n)p - \frac{1-a}{u_3}] u_3^{m+n-1} h(\theta_1, u_3 | \underline{x}, \underline{y}) du_3 \ ,$$

where

$$h(\theta_1, u_3 | \underline{x}, \underline{y}) = (\prod_{i=1}^{m} x_i \prod_{j=1}^{n} y_j)^{pu_3 - 1} [\theta_1 \sum_{i=1}^{m} x_i^{u_3} + \sum_{j=1}^{n} y_j^{u_3}]^{-(m+n)p} \ .$$

References

1. Bhattacharyya, G. K. and Johnson, R. A.(1974). Estimation of Reliability in a Multicomponent Stress-Strength Model. *J.Amer.Statist. Assoc.*, 69, 966-970.
2. Cox, D. R. and Reid, N.(1987). Orthogonal Parameter and Approximate Conditional Inference(with discussion), *J. Royal Statist. Soc. (Ser. B.)*}, 49, 1-39.
3. Datta, G. S. and Ghosh, M.(1995). "Some Remarks on Noninformative Prior." *J. Amer. Statist. Assoc.,* 90, No.432, 1357-1363.
4. Draper, N. R. and Guttman, I.(1978). Bayesian Analysis of Reliability in Multicomponent Stress-Strength Models. *Commu. Statist.-Theor. Math.*, A7(5), 441-445.
5. Guttman, I., Reiser, B., Bhattacharyya, G. K., and Johnson, R. A.(1990). Bayesian Inference for Stress-Strength Models with Explanatory Variables.
6. *Gujarat Statist. Rev.-Professor Khatri Memorial Volume,* 53-67.

   Guttman, I. and Papandonatos, G. D.(1997). A Bayesian Approach to a Reliability Problem : Theory, Analysis and Interesting Numerics. *Canad. J. Statist.,* Vol 25, No 2, 143-158.
7. Peers, H . W.(1956). On Confidence Sets and Bayesian Probability Points in the Case of Several Parameters, *J. Royal Statist. Soc. Ser. B,* 27, 9-16.
8. Reiser, B. and Guttman, I.(1989). Sample Size Choice for Reliability Verification in Stress-Strength Models. *J. Statist.,* 17, 253-259.
9. Thomson, R. D. and Basu, A. P.(1993). Bayesian Reliability of

Stress-Strength System. Advances in Reliability (A. Basu, ed.), *North-Holland*, Amsterdam, 411-421.

10. Tibshirani, R.(1989). Noninformative Priors for One Parameter of Many. *Biometrika,* 76, 604-608.
11. Welch, B. L. and Peers, H. W.(1963). On Formulae for Confidence Points Based on Intergrals of weighted Likelihoods. *J. Royal Statist. Soc. Ser. B,* 25, 318-329.