Proceedings of Korean Data And Information Science Society,
October 28-29, 2005, pp. 13-26

Reliability Estimation for the Exponential Distribution under Multiply Type-II Censoring

Suk-Bok Kang[1] · Sang-Ki Lee[2] · Hui-Taeg Choi[3]

Abstract

In this paper, we derive the approximate maximum likelihood estimators of the scale parameter and location parameter of the exponential distribution based on multiply Type-II censored samples. We compare the proposed estimators in the sense of the mean squared error for various censored samples. We also obtain the approximate maximum likelihood estimator (AMLE) of the reliability function by using the proposed estimators. And then we compare the proposed estimators in the sense of the mean squared error.

Keywords : Approximate maximum likelihood estimator, Exponential distribution, Multiply Type-II censored sample, Reliability

1. Introduction

The exponential distribution occupies an important position in life testing and reliability problems, especially in the area of industrial life testing. The failure time X is said to follow two-parameter exponential distribution if the probability density function (pdf) of X is of the form

$$f(x;\theta,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{x-\theta}{\sigma}\right),\quad x>\theta,\ \sigma>0 \tag{1.1}$$

and the cumulative distribution function (cdf)

$$F(x;\theta,\sigma)=1-\exp\left(-\frac{x-\theta}{\sigma}\right),\quad x>\theta,\ \sigma>0\,. \tag{1.2}$$

The data for estimating the scale and the location parameters of the two-parameter exponential distribution is usually obtained through Type-II censored sampling scheme. The problem of estimating parameters have been investigated by many authors.

Especially, the approximate maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution.

---

1) Professor, Department of Statistics, Yeungnam University, Gyeongsan, 712-749, Korea
E-mail : sbkang@yu.ac.kr

2) Graduate student, Department of Statistics, Yeungnam University, Gyeongsan, 712-749, Korea

3) Graduate student, Department of Statistics, Yeungnam University, Gyeongsan, 712-749, Korea

Kang and Cho (1997) studied estimation for the exponential distribution under general progressive Type-II censored samples. Escobar and Meeker (2001) showed that the Fisher information matrices for Type-I and Type-II censoring are asymptotically equivalent. Lin and Balakrishnan (2003) developed the exact prediction intervals for the failure times from the one-parameter and two-parameter exponential distributions based on doubly Type-II censored samples and they presented a computational algorithm for the determination of the exact percentage points of the pivotal quantities used in the construction of the prediction intervals.

Multiply Type-II censoring is a more general than Type-II censoring, but mathematically and numerically much more complicated censoring scheme. Balasubramanian and Balakrishnan (1992), Upadhyay et al. (1996) considered estimation for the exponential distribution under multiply Type-II censoring. Recently, Kang (2003) proposed the AMLE of the location and the scale parameters of the two-parameter exponential distribution with multiply Type-II censoring.

In life testing research we are more concerned with a quantitative measure of the reliability of an item or a device we are interested in. Engelhardt and Bain (1978) proposed the tolerance limits and confidence limits on the reliability for the two-parameter exponential distribution. Chiou (1987) proposed a preliminary test estimator for the reliability of an exponential life-testing model.

In this paper, we consider unbiased estimator of the location parameter, and we derive estimator of the location parameter by minimizing the mean squared error (MSE) of the linear combination of some available order statistics. we also propose the AMLEs of the scale parameter σ of the two-parameter exponential distribution with multiply Type-II censoring. The scale parameter is estimated by approximate maximum likelihood estimation method using two different types Taylor series expansions. We also obtain the AMLE of the reliability function by using the proposed estimators. And then we compare the proposed estimators in the sense of the MSE.

2. Approximate Maximum Likelihood Estimators

Let

$$X_{a_1:n} \le X_{a_2:n} \le \cdots \le X_{a_s:n} \tag{2.1}$$

be the available multiply Type-II censored sample from the exponential distribution with pdf (1.1), where

$$1 \le a_1 < a_2 < \cdots < a_s \le n.$$

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) is given by

$$
\begin{aligned}
L &= \frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!} \prod_{j=1}^{s+1}[F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1} \frac{1}{\sigma^s} \prod_{j=1}^{s} f(Z_{a_j:n}) \\
&= \frac{1}{\sigma^s} \frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!} [F(Z_{a_j:n})]^{a_1 - 1}[1 - F(Z_{a_s:n})]^{n - a_s} \\
&\quad \times \prod_{j=1}^{s} f(Z_{a_j:n}) \prod_{j=2}^{s}[F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1}
\end{aligned}
\tag{2.2}
$$

where $Z_{i:n} = (X_{i:n} - \theta)/\sigma$, and $f(z) = e^{-z}$ and $F(z) = 1 - e^{-z}$ are pdf and cdf of the standard exponential distribution, respectively.

2.1 Estimation of the location parameter

We consider some estimators of the location parameter θ.

The following estimator is well known estimator of the location parameter.

$$\widehat{\theta}_1 = X_{a_1:n}. \tag{2.1.1}$$

But $\widehat{\theta}_1$ always overestimate the location parameter θ, so we consider another unbiased estimator which is linear combination of the minimum available order statistic as follows;

$$\widehat{\theta}_2 = c_1 X_{a_1:n} + c_2 X_{a_2:n}. \tag{2.1.2}$$

The expectation of $\widehat{\theta}_2$ is given by

$$E(\widehat{\theta}_2) = (c_1 + c_2)\theta + \sigma\left[c_1 \sum_{j=1}^{a_1}(n-j+1)^{-1} + c_2 \sum_{j=1}^{a_2}(n-j+1)^{-1}\right] \tag{2.1.3}$$

where c_1 and c_2 are constants.

From equations (2.1.2) and (2.1.3), we can easily obtain an unbiased estimator of the location parameter as follows;

$$\widehat{\theta}_2 = \frac{1}{h(a_2) - h(a_1)}[h(a_2) X_{a_1:n} - h(a_1) X_{a_2:n}] \tag{2.1.4}$$

where

$$h(a) = \sum_{j=1}^{a}(n-j+1)^{-1}.$$

Also we can derive the other estimator by minimizing the MSE among the class of estimators of the form $[1-(s-1)d]X_{a_1:n} + d\sum_{j=2}^{s} X_{a_j:n}$ where d is constant.

Let

$$\widehat{\theta}_3 = [1-(s-1)d]\,X_{a_1:n} + d\sum_{j=2}^{s} X_{a_j:n}. \tag{2.1.5}$$

The MSE of $\widehat{\theta}_3$ is given by

$$\begin{aligned} \mathrm{MSE}(\widehat{\theta}_3) = & \Big[[1-(s-1)d]^2 [g(a_1) + h^2(a_1)] \\ & + d^2\Big\{ \sum_{j=2}^{s} g(a_j) + 2\sum_{j=2}^{s-1}(s-j)\,g(a_j) + \Big(\sum_{j=2}^{s} h(a_j)\Big)^2 \Big\} \\ & + 2d[1-(s-1)d]\Big\{(s-1)g(a_1) + h(a_1)\sum_{j=2}^{s} h(a_j)\Big\}\Big]\sigma^2. \end{aligned} \tag{2.1.6}$$

From equation (2.1.6), we can also obtain the constant d which minimize $\mathrm{MSE}(\widehat{\theta}_3)$ by differentiation. So we propose the estimator of the location parameter as follows;

$$\widehat{\theta}_3 = [1-(s-1)d]\,X_{a_1:n} + d\sum_{j=2}^{s} X_{a_j:n} \tag{2.1.7}$$

where

$$d = \frac{h(a_1)\Big[(s-1)\,h(a_1) - \sum_{j=2}^{s} h(a_j)\Big]}{v}$$

$$\begin{aligned} v = & (s-1)^2\big[h^2(a_1) - g(a_1)\big] + \sum_{j=2}^{s} g(a_j) + 2\sum_{j=1}^{s-1}(s-j)\,g(a_j) + \Big[\sum_{j=2}^{s} h(a_j)\Big]^2 \\ & - 2(s-1)\,h(a_1)\sum_{j=2}^{s} h(a_j) \end{aligned}$$

and

$$g(a) = \sum_{j=1}^{a}(n-j+1)^{-2}.$$

2.2 Estimation of the scale parameter

We consider the estimation of the scale parameter σ.
From equation (2.2), we can obtain the following likelihood equation for σ.

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} = & -\frac{1}{\sigma}\Bigg[s + (a_1 - 1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} - (n-a_s)\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})} Z_{a_s:n} \\ & - \sum_{j=1}^{s} Z_{a_j:n} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})}\Bigg] \\ = & \ 0. \end{aligned} \tag{2.2.1}$$

The equation (2.2.1) does not admit an explicit solution for σ. But we can expand the following functions

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})},\quad \frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})},\quad \frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}$$

in Taylor series around the points ξ_{a_1} and $(\xi_{a_j},\ \xi_{a_{j-1}})$ respectively, where

$$\xi_{a_1}=F^{-1}(p_{a_1})=-\ln(1-p_{a_1})$$
$$\xi_{a_j}=F^{-1}(p_{a_j})=-\ln(1-p_{a_j})$$
$$\xi_{a_{j-1}}=F^{-1}(p_{a_{j-1}})=-\ln(1-p_{a_{j-1}})$$

and

$$p_i=\frac{i}{n+1}.$$

Therefore, we can approximate these functions by

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}\simeq \alpha_1+\beta_1 Z_{a_1:n} \tag{2.2.2}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq \alpha_{1j}+\beta_{1j}Z_{a_j:n}+\gamma_{1j}Z_{a_{j-1}:n} \tag{2.2.3}$$

and

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq \alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n} \tag{2.2.4}$$

where

$$\alpha_1=\frac{f(\xi_{a_1})}{p_{a_1}}\left[1+\xi_{a_1}+\frac{f(\xi_{a_1})}{p_{a_1}}\,\xi_{a_1}\right]$$
$$\beta_1=-\frac{f(\xi_{a_1})}{p_{a_1}}\left[1+\frac{f(\xi_{a_1})}{p_{a_1}}\right]$$
$$\alpha_{1j}=\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1+\xi_{a_j}+\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$
$$\beta_{1j}=-\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1+\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\right)$$
$$\gamma_{1j}=\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}$$
$$\alpha_{2j}=\frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\left(1+\xi_{a_{j-1}}+\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$
$$\beta_{2j}=-\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}=-\gamma_{1j}$$

and

$$\gamma_{2j} = -\frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}\left(1 - \frac{f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}\right).$$

By substituting equations (2.2.2), (2.2.3), and (2.2.4) into equation (2.2.1), we obtain the approximate likelihood equation of (2.2.1) as

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma} \\ &= -\frac{1}{\sigma}\Big[s + (a_1 - 1)(\alpha_1 + \beta_1 Z_{a_1:n}) Z_{a_1:n} - (n - a_s) Z_{a_s:n} \\ &\quad + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\left[(\alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n}) Z_{a_j:n} \right. \\ &\quad \left. - (\alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n}) Z_{a_{j-1}:n} \right] - \sum_{j=1}^{s} Z_{a_j:n} \Big] \\ &= 0 .\end{aligned} \tag{2.2.5}$$

Equation (2.2.5) is quadratic in σ as follows;

$$s\sigma^2 + B_{1i}\sigma + C_{1i} = 0 \tag{2.2.6}$$

where

$$\begin{aligned} B_{1i} = &\ (a_1 - 1)\alpha_1 X_{a_1:n} - (n - a_s) X_{a_s:n} - \sum_{j=1}^{s} X_{a_j:n} \\ &+ \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{1j} X_{a_j:n} - \alpha_{2j} X_{a_{j-1}:n}) \\ &- \Big[(a_1 - 1)\alpha_1 - (n - a_s) - s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{1j} - \alpha_{2j}) \Big] \widehat{\theta}_i \\ C_{1i} = &\ \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\Big\{ \beta_{1j}(X_{a_j:n} - \widehat{\theta}_i)^2 + 2\gamma_{1j}(X_{a_j:n} - \widehat{\theta}_i)(X_{a_{j-1}:n} - \widehat{\theta}_i) \\ &- \gamma_{2j}(X_{a_{j-1}:n} - \widehat{\theta}_i)^2 \Big\} + (a_1 - 1)\beta_1 (X_{a_1:n} - \widehat{\theta}_i)^2. \end{aligned}$$

and $\widehat{\theta}_0 = \theta_0$ is known location parameter.

Upon solving equation (2.2.6) for σ, we first derive an AMLE of σ as

$$\widehat{\sigma}_{1i} = \frac{-B_{1i} + \sqrt{B_{1i}^2 - 4sC_{1i}}}{2s}, \quad i = 0, 1, 2, 3. \tag{2.2.7}$$

Second, we can expand the functions.

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} \text{ and } \frac{f(Z_{a_j:n}) Z_{a_j:n} - f(Z_{a_{j-1}:n}) Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})}$$

Therefore, we can approximate these functions by

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} \simeq \alpha_2 + \beta_2 Z_{a_1:n} \tag{2.2.8}$$

and

$$\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq \alpha_j+\beta_j Z_{a_j:n}+\gamma_j Z_{a_{j-1}:n} \tag{2.2.9}$$

where

$$\alpha_2=\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\left[\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}+\xi_{a_1}\right]$$

$$\beta_2=\frac{f(\xi_{a_1})}{p_{a_1}}\left[1-\xi_{a_1}-\frac{f(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\alpha_j=\frac{f(\xi_{a_j})\xi_{a_j}^{\ 2}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}^{\ 2}}{p_{a_j}-p_{a_{j-1}}}+\left(\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)^2$$

$$\beta_j=\frac{f(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\left(1-\xi_{a_j}-\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

and

$$\gamma_j=-\frac{f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\left(1-\xi_{a_{j-1}}-\frac{f(\xi_{a_j})\xi_{a_j}-f(\xi_{a_{j-1}})\xi_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\right)$$

in Taylor series around the points ξ_{a_1} and $(\xi_{a_j},\ \xi_{a_{j-1}})$ respectively.

By substituting equations (2.2.8) and (2.2.9) into equation (2.2.1), we obtain the secondly approximate likelihood equation of (2.2.1) as

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma}\\ &=-\frac{1}{\sigma}\Big[s+(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})-(n-a_s)Z_{a_s:n}\\ &\quad+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_j+\beta_j Z_{a_j:n}+\gamma_j Z_{a_{j-1}:n})-\sum_{j=1}^{s}Z_{a_j:n}\Big]\\ &=0.\end{aligned} \tag{2.2.10}$$

From equation (2.2.10), we can derive more simple estimator of σ which is linear function of the available order statistics as

$$\hat{\sigma}_{2i}=-\frac{B_{2i}}{A_2} \tag{2.2.11}$$

where

$$A_2=s+(a_1-1)\alpha_2+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_j$$

and

$$\begin{aligned}B_{2i}=\ &(a_1-1)\beta_2 X_{a_1:n}-(n-a_s)X_{a_s:n}-\sum_{j=1}^{s}X_{a_j:n}\\ &+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_j X_{a_j:n}+\gamma_j X_{a_{j-1}:n})\\ &-\Big[(a_1-1)\beta_2-(n-a_s)-s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_j+\gamma_j)\Big]\hat{\theta}_i.\end{aligned}$$

Third, Johnson et al. (1994) derived the best linear unbiased estimator of the scale parameter σ when the location parameter θ is known as follows;

$$\hat{\sigma}=\left[\sum_{j=1}^{k}\left(\frac{w_{1j}}{w_{2j}}-\frac{w_{1,j+1}}{w_{2,j+1}}\right)X_{a_j:n}-\frac{w_{11}}{w_{21}}\theta\right]\left[\sum_{j=1}^{k}\frac{w_{1j}}{w_{2j}}\right]^{-1}$$

where

$$w_{mj}=\sum_{i=a_{j-1}}^{a_j-1}(n-i)^{-m}$$

and $\frac{w_{10}}{w_{20}}$ and $\frac{w_{1,k+1}}{w_{2,k+1}}$ defined to be zero.

We can use the estimator $\hat{\sigma}$ when the location parameter θ is unknown as follows;

$$\hat{\sigma}_{3i}=\left[\sum_{j=1}^{k}\left(\frac{w_{1j}}{w_{2j}}-\frac{w_{1,j+1}}{w_{2,j+1}}\right)X_{a_j:n}-\frac{w_{11}}{w_{21}}\hat{\theta}_i\right]\left[\sum_{j=1}^{k}\frac{w_{1j}}{w_{2j}}\right]^{-1}. \qquad (2.2.12)$$

2.3 Estimation of the Reliability

The exponential distribution plays an important role in the field of reliability.

If the failure-time X of a component or system is exponentially distributed with scale parameter σ and location parameter θ, then the reliability function is

$$R(t)=1-F(t)=P[X>t]=\exp\left(-\frac{t-\theta}{\sigma}\right), \quad t>\theta,\ \sigma>0. \qquad (2.3.1)$$

So the probability of survival of an item until time t(called the reliability at time t) is $R(t)$.

Since the scale parameter σ and the location parameter θ are unknown, we now consider the estimation of $R(t)$. If θ is known, say $\theta=\theta_o$ (which we may take to be zero without any loss of generality), then we obtain the single parameter exponential distribution with mean σ. However, in most cases θ is unknown and one of the objectives of life testing experiments would be estimate θ.

We also consider the multiply Type-II censored data as in (2.1). So we now propose the AMLEs of the reliability function $R(t)$ by using the proposed AMLEs $\hat{\sigma}_{kl}$ and $\hat{\theta}_l$ that can be used for multiply Type-II censored sample as follows.

$$\hat{R}_{lk}(t)=\exp\left(-\frac{t-\hat{\theta}_l}{\hat{\sigma}_{kl}}\right), \quad l=0,1,2,3, \quad k=1,2,3. \qquad (2.3.2)$$

From equations (2.1.1), (2.1.4), (2.1.7), (2.2.7), (2.2.11), and (2.2.12), the mean squared errors of these estimators are simulated by Monte Carlo method for sample size $n=20,50$ and various choices of censoring. The simulation procedure is repeated 10,000 times in multiply Type-II censored samples. From Table 1, the estimators $\hat{\theta}_3$ is more efficient than the other estimators in the sense of MSE. From Table 2, when the location parameter is known, the estimators $\hat{\sigma}_{20}$ and $\hat{\sigma}_{30}$ of the scale parameter are generally more efficient than the estimator $\hat{\sigma}_{10}$ in the sense of MSE. When the location parameter is unknown, the estimators $\hat{\sigma}_{1i}$, $\hat{\sigma}_{2i}$, and $\hat{\sigma}_{3i}$ are generally efficient with using the estimator $\hat{\theta}_1$ of the location parameter θ. From equation (2.3.2), the mean squared errors of these estimators are simulated by Monte Carlo method for sample size $n=20$, $t=0.5, 1.0$ and various choices of censoring. From Table 3, When the location parameter θ is unknown, the estimator $\hat{R}_{3k}(t)$ is generally more efficient than the other estimators $\hat{R}_{lk}(t)$. As expected, the MSE of $\hat{R}_{lk}(t)$ generally increases as k increases. And the MSE of $\hat{R}_{lk}(t)$ is a little small when the location parameter is known.

Table 1. The relative mean squared errors for the estimators of the location parameter θ.

n	k	a_j	MSE			n	k	a_j	MSE		
			$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
20	0	1~20	0.0048	0.0050	0.0025	50	0	1~50	0.0008	0.0008	0.0004
	2	1~18	0.0048	0.0050	0.0025		2	1~48	0.0008	0.0008	0.0004
		3~20	0.0334	0.0348	0.0099			3~50	0.0025	0.0024	0.0009
		2~19	0.0158	0.0159	0.0059			2~49	0.0025	0.0024	0.0009
	5	3~17	0.0334	0.0348	0.0103		5	3~47	0.0085	0.0086	0.0019
		4~18	0.0591	0.0604	0.0153			4~48	0.0085	0.0086	0.0019
		2~6 10~19	0.0158	0.0159	0.0059			2~6 10~19 21~50	0.0025	0.0024	0.0009
	6	4~17	0.0591	0.0604	0.0156		6	4~47	0.0025	0.0024	0.0009
		1 2 6~9 12~15 17~20	0.0048	0.0050	0.0025			1 2 6~9 12~15 17~50	0.0008	0.0008	0.0004

Table 2. The relative mean squared errors for the estimators of the scale parameter σ.

n	k	a_j	MSE							
			$\hat{\sigma}_{10}$	$\hat{\sigma}_{11}$	$\hat{\sigma}_{12}$	$\hat{\sigma}_{13}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{21}$	$\hat{\sigma}_{22}$	$\hat{\sigma}_{23}$
20	0	1～20	0.0508	0.0505	0.0535	0.0532	0.0508	0.0505	0.0535	0.0532
	2	1～18	0.0567	0.0562	0.0600	0.0596	0.0567	0.0562	0.0600	0.0596
		3～20	0.0556	0.0565	0.0602	0.0593	0.0508	0.0653	0.0602	0.0593
		2～19	0.0577	0.0565	0.0602	0.0595	0.0537	0.0588	0.0602	0.0595
	5	3～17	0.0661	0.0677	0.0734	0.0717	0.0599	0.0797	0.0734	0.0717
		4～18	0.0629	0.0680	0.0737	0.0717	0.0567	0.0926	0.0737	0.0716
		2～6 10～19	0.0651	0.0599	0.0664	0.0655	0.0537	0.0588	0.0603	0.0596
	6	4～17	0.0666	0.0725	0.0793	0.0767	0.0599	0.1002	0.0792	0.0767
		1 2 6～9 12～15 17～20	0.0705	0.0630	0.0734	0.0728	0.0511	0.0507	0.0538	0.0536
50	0	1～50	0.0196	0.0196	0.0199	0.0199	0.0196	0.0196	0.0199	0.0199
	2	1～48	0.0208	0.0204	0.0207	0.0207	0.0208	0.0204	0.0207	0.0207
		3～50	0.0203	0.0205	0.0208	0.0208	0.0196	0.0222	0.0208	0.0208
		2～49	0.0207	0.0204	0.0207	0.0207	0.0199	0.0209	0.0207	0.0207
	5	3～47	0.0214	0.0218	0.0222	0.0221	0.0203	0.0238	0.0222	0.0221
		4～48	0.0219	0.0218	0.0221	0.0221	0.0208	0.0259	0.0221	0.0221
		2～6 10～19 21～50	0.0225	0.0213	0.0224	0.0223	0.0196	0.0206	0.0204	0.0204
	6	4～47	0.0226	0.0223	0.0227	0.0226	0.0196	0.0266	0.0227	0.0226
		1 2 6～9 12～15 17～50	0.0228	0.0216	0.0232	0.0231	0.0196	0.0196	0.0199	0.0199

Table 2. (continued)

n	k	a_j	MSE			
			$\hat{\sigma}_{30}$	$\hat{\sigma}_{31}$	$\hat{\sigma}_{32}$	$\hat{\sigma}_{33}$
20	0	1～20	0.0506	0.0503	0.0531	0.0529
	2	1～18	0.0568	0.0566	0.0601	0.0597
		3～20	0.0506	0.0651	0.0595	0.0586
		2～19	0.0531	0.0584	0.0595	0.0589
	5	3～17	0.0605	0.0803	0.0735	0.0719
		4～18	0.0569	0.0930	0.0735	0.0715
		2～6 10～19	0.0534	0.0586	0.0598	0.0592
	6	4～17	0.0605	0.1008	0.0796	0.0772
		1 2 6～9 12～15 17～20	0.0506	0.0504	0.0532	0.0529
50	0	1～50	0.0200	0.0200	0.0204	0.0204
	2	1～48	0.0208	0.0208	0.0213	0.0213
		3～50	0.0200	0.0224	0.0213	0.0213
		2～49	0.0203	0.0213	0.0214	0.0213
	5	3～47	0.0212	0.0239	0.0227	0.0226
		4～48	0.0208	0.0260	0.0228	0.0227
		2～6 10～19 21～50	0.0200	0.0209	0.0210	0.0209
	6	4～47	0.0212	0.0266	0.0233	0.0232
		1 2 6～9 12～15 17～50	0.0200	0.0200	0.0204	0.0204

Table 3. The relative mean squared errors for the estimators of the reliability function $R(t)$.

n	k	a_j	MSE of $\widehat{R}_{lk}(t)$ ($t=0.5$)				MSE of $\widehat{R}_{lk}(t)$ ($t=1.0$)			
			$\widehat{R}_{01}(t)$	$\widehat{R}_{11}(t)$	$\widehat{R}_{21}(t)$	$\widehat{R}_{31}(t)$	$\widehat{R}_{01}(t)$	$\widehat{R}_{11}(t)$	$\widehat{R}_{21}(t)$	$\widehat{R}_{31}(t)$
20	0	1~20	0.0049	0.0061	0.0055	0.0053	0.0065	0.0072	0.0066	0.0066
	2	1~18	0.0056	0.0067	0.0061	0.0059	0.0073	0.0081	0.0074	0.0074
		3~20	0.0048	0.0159	0.0083	0.0065	0.0066	0.0103	0.0073	0.0068
		2~19	0.0051	0.0093	0.0066	0.0061	0.0069	0.0086	0.0072	0.0071
	5	3~17	0.0058	0.0162	0.0088	0.0074	0.0078	0.0115	0.0086	0.0081
		4~18	0.0054	0.0286	0.0121	0.0082	0.0073	0.0146	0.0088	0.0078
		2~6 10~19	0.0051	0.0099	0.0067	0.0060	0.0072	0.0091	0.0076	0.0072
	6	4~17	0.0057	0.0285	0.0122	0.0085	0.0078	0.0149	0.0092	0.0083
		1 2 6~9 12~15 17~20	0.0049	0.0066	0.0055	0.0053	0.0072	0.0080	0.0073	0.0072

n	k	a_j	MSE of $\widehat{R}_{lk}(t)$ ($t=0.5$)				MSE of $\widehat{R}_{lk}(t)$ ($t=1.0$)			
			$\widehat{R}_{02}(t)$	$\widehat{R}_{12}(t)$	$\widehat{R}_{22}(t)$	$\widehat{R}_{32}(t)$	$\widehat{R}_{02}(t)$	$\widehat{R}_{12}(t)$	$\widehat{R}_{22}(t)$	$\widehat{R}_{32}(t)$
20	0	1~20	0.0049	0.0061	0.0055	0.0053	0.0065	0.0072	0.0066	0.0066
	2	1~18	0.0056	0.0067	0.0061	0.0059	0.0073	0.0081	0.0074	0.0074
		3~20	0.0049	0.0135	0.0087	0.0065	0.0065	0.0092	0.0070	0.0068
		2~19	0.0052	0.0088	0.0069	0.0061	0.0069	0.0085	0.0071	0.0071
	5	3~17	0.0059	0.0140	0.0091	0.0074	0.0077	0.0110	0.0083	0.0081
		4~18	0.0056	0.0236	0.0121	0.0082	0.0073	0.0120	0.0082	0.0078
		2~6 10~19	0.0052	0.0088	0.0069	0.0061	0.0069	0.0086	0.0072	0.0071
	6	4~17	0.0059	0.0237	0.0122	0.0085	0.0077	0.0128	0.0087	0.0083
		1 2 6~9 12~15 17~20	0.0049	0.0061	0.0055	0.0053	0.0065	0.0072	0.0066	0.0066

Table 2. (continued)

n	k	a_j	MSE of $\hat{R}_{lk}(t)$				MSE of $\hat{R}_{lk}(t)$			
			$\hat{R}_{03}(t)$	$\hat{R}_{13}(t)$	$\hat{R}_{23}(t)$	$\hat{R}_{33}(t)$	$\hat{R}_{03}(t)$	$\hat{R}_{13}(t)$	$\hat{R}_{23}(t)$	$\hat{R}_{33}(t)$
20	0	1～20	0.0049	0.0061	0.0055	0.0053	0.0065	0.0072	0.0066	0.0066
	2	1～18	0.0056	0.0067	0.0061	0.0059	0.0073	0.0081	0.0074	0.0074
		3～20	0.0049	0.0135	0.0087	0.0065	0.0065	0.0092	0.0070	0.0068
		2～19	0.0052	0.0088	0.0069	0.0061	0.0069	0.0085	0.0071	0.0071
	5	3～17	0.0059	0.0140	0.0091	0.0074	0.0077	0.0110	0.0083	0.0081
		4～18	0.0056	0.0236	0.0121	0.0082	0.0073	0.0120	0.0082	0.0078
		2～6 10～19	0.0052	0.0088	0.0069	0.0061	0.0069	0.0086	0.0072	0.0071
	6	4～17	0.0059	0.0237	0.0122	0.0085	0.0077	0.0128	0.0087	0.0083
		1 2 6～9 12～15 17～20	0.0049	0.0061	0.0055	0.0053	0.0065	0.0072	0.0066	0.0066

References

1. Balakrishnan, N. (1989). Approximate MLE of the scale parameter of the Rayleigh distribution with censoring, *IEEE Transactions on Reliability*, 38, 355-357.

2. Balasubramanian, K. and Balakrishnan, N. (1992). Estimation for one-parameter and two-parameter exponential distributions under multiple Type-II censoring, *Statistische Hefte*, 33, 203-216.

3. Chiou, P. (1987). A preliminary test estimator of reliability in a life-testing model, *IEEE Transactions on Reliability*, 36, 408-410.

4. Engelhardt M. and Bain L. J. (1978). Tolerance limits and confidence limits on reliability for the two-parameter exponential distribution, *Technometrics*, 20, 37-39.

5. Escobar, L. A. and Meeker, W. Q. (2001). The asymptotic equivalence of the Fisher information matrices for Type-I and Type-II censored data from location-scale families, *Communications in Statistics-Theory and Methods*, 30, 2211-2225.

6. Johnson, N. L., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, John Wiley & Sons, Inc.

7. Kang, S. B. (2003). Approximate MLEs for exponential distribution under multiple Type-II censoring, *Journal of the Korean Data & Information Science Society*, 14, 983-988.

8. Kang, S. B. and Cho, Y. S. (1997). Estimation for exponential distribution under general progressive Type-II censored samples, *Journal of Statistical Theory & Methods*, 8, 239-245.

9. Lin, C. T. and Balakrishnan, N. (2003). Exact prediction intervals for exponential distributions based on doubly Type-II censored samples, *Journal of Applied Statistics*, 30, 783-801.

10. Upadhyay, S. K., Singh, U., and Shastri, V. (1996). Estimation of exponential parameters under multiply Type-II censoring, *Communications in Statistics-Simulation and Computation*, 25, 801-815.